Topology and its Applications 158 (2011) 882–903

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Lax algebras via initial monad morphisms: APP, TOP, MET and ORD

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ARTICLE INFO

MSC:
18B10
18C15
18C20
54A20
54E99

Keywords:
Monad
Lax extension
Monad morphism
Lax algebra
Prime functional ideal
Ultrafilter
Approach space
Metric space
Topological space
Ordered space

ABSTRACT

This paper contributes to the algebraization of topology via the theory of monads and lax extensions of monads and their associated lax algebras (see Barr (1970) [1], Clementino and Hofmann (2003) [2], Clementino, Hofmann and Tholen (2004) [4], Clementino and Tholen (2003) [5], Lowen and Vroegrijk (2008) [11], Manes (1974) [12], Seal (2005) [14]). We construct a monad $\mathbb{P}$, a lax extension $\overline{\mathbb{P}}$ and monad morphisms into $\mathbb{P}$ from the most important monads as studied in the aforementioned papers such that their lax extensions and their associated categories of lax algebras can be derived from the extension $\overline{\mathbb{P}}$ by initial lifts via these monad morphisms. This provides us with a completely unified way to obtain the categories Top, App, Met and Ord without the necessity to leave the realm of Rel as was previously required in Clementino and Hofmann (2003) [2], Clementino, Hofmann and Tholen (2004) [4] and Clementino and Tholen (2003) [5] in particular in order to obtain App and Met.

1. Introduction

In [12] Manes proved that compact Hausdorff spaces can be obtained as the Eilenberg–Moore algebras of the ultrafilter monad. Later Barr [1] showed that by going from the Set-setting to Rel and relaxing the axioms on the monad and the algebras derived it was possible to obtain all topological spaces as lax algebras of a suitable lax extension of the ultrafilter monad to Rel. Following a suggestion of Janelidze and inspired by [8], later, Clementino and Hofmann and Clementino, Hofmann and Tholen proved that by further extending the setting of Rel to a numeric version nRel they could capture pseudo-quasi-metric spaces and approach spaces [9] as categories of lax algebras [2,4]. This line of investigation was pursued in further work of Clementino, Hofmann and Tholen [5,3] and Seal and Schubert [14,15]. In [10,11] Lowen, Vroegrijk and Van Olmen introduced the prime functional ideal monad and showed that it is possible to obtain these results without the necessity to go from Rel to nRel by using a more general notion of extension of a monad as introduced by Seal in [14]. It is this more general notion of extension which we use in this paper. We construct a supermonad $\mathbb{P} = (\mathsf{P}, e, m)$, define a lax extension $\overline{\mathbb{P}}$ and construct various monad morphisms into $\mathbb{P}$ from the most important monads (identity, ultrafilter, prime functional ideal and a new monad to obtain metric spaces) studied in [1,2,4,5,11,12,14] such that their lax extensions can be derived from the extension $\overline{\mathbb{P}}$ by initial lifts via these monad morphisms.

In the following diagram of monads, $\mathbb{P}$ stands for the supermonad defined in Section 3, $\mathbb{I}$ is the identity monad, $\boldsymbol{\beta}$ the ultrafilter monad, $\boldsymbol{\pi}$ the prime functional ideal monad (see Section 4), used to obtain approach spaces in [10,11], and $\mathbb{D}$

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0166-8641/$ – see front matter
doi:10.1016/j.topol.2011.01.008

is the monad defined in Section 5, used to obtain pseudo-quasi-metric spaces. We define all arrows and show that they are monad morphisms. We identify $\mathsf{P}_D X$, $\mathsf{P}_\pi X$, $\mathsf{P}_\beta X$ and $\mathsf{P}_\mathsf{I} X$ as the images of respectively $\mathsf{D}X$, πX, βX and $\mathsf{I}X$ under monad morphisms into $\mathbb{P}$, proving that the four corresponding horizontal arrows are actually monad isomorphisms, thus embedding the monads $\mathbb{I}$, $\boldsymbol{\beta}$, $\boldsymbol{\pi}$ and $\mathbb{D}$ into $\mathbb{P}$.

$$
\begin{tikzcd}
 & \mathbb{D} \arrow[dr] \arrow[rrr] & & & \mathbb{P}_D \arrow[dr] & & \\
\mathbb{I} \arrow[ur] \arrow[rr] \arrow[dr] \arrow[rrr, bend left] & & \boldsymbol{\pi} \arrow[dl, bend left, "\iota"] \arrow[rrr, bend right] & \mathbb{P}_I \arrow[ur] \arrow[rr] \arrow[dr] & & \mathbb{P}_\pi \arrow[r] \arrow[dl, bend left, "\iota'"] & \mathbb{P} \\
 & \boldsymbol{\beta} \arrow[ur, bend left, "\omega"] \arrow[rrr] & & & \mathbb{P}_\beta \arrow[ur, bend left, "\omega'"] & &
\end{tikzcd}
$$

The algebraic part, namely the description of the Eilenberg–Moore algebras associated with these monads will be published elsewhere. In this paper we concentrate on lax algebras. In Section 7, we define a suitable lax extension $\overline{\mathbb{P}}$ of $\mathbb{P}$ and construct all lax extensions derived via initiality along the monad morphisms in the above diagram. Finally, in Section 8, we describe the associated categories of lax algebras. This gives us the category Ord of ordered spaces as $\mathsf{Alg}(\overline{\mathbb{I}})$, the category pq-Met^∞ of pseudo-quasi-metric spaces obtained as a reflective subcategory of $\mathsf{Alg}(\overline{\mathbb{D}})$, the category Top of topological spaces as $\mathsf{Alg}(\overline{\boldsymbol{\beta}})$ and finally the category App of approach spaces as $\mathsf{Alg}(\overline{\boldsymbol{\pi}})$. The adjunction $\omega \vdash \iota$ on the level of the monad morphisms gives rise to the coreflective embedding of Top in App. In all cases the "isomorphic" second diamond in the above diagram gives rise to new descriptions of these categories via the associated categories of lax algebras.

2. The functional power monad

Let $[0,\infty]$ be endowed with usual order $\leqslant$ and addition $+$. For α and γ in $[0,\infty]$ we put $\alpha \ominus \gamma = (\alpha - \gamma) \vee 0$, where $\infty \ominus \infty = 0$ and $\infty \ominus \gamma = \infty$ and $\alpha \ominus \infty = 0$ when $\alpha, \gamma \in [0,\infty[$. For any set X let BX stand for the set of all bounded functions in $[0,\infty]^X$. For $\alpha \in [0,\infty]$ the corresponding constant function in BX is again denoted by α. Further let $\mathsf{P}X$ stand for the set of all functions

$$\mathfrak{f} : BX \to [0,\infty]$$

that satisfy the following conditions:

(P1) $\mathfrak{f}$ is order-preserving,
(P2) for any $\mu \in BX$ and α constant: $\mathfrak{f}(\mu \ominus \alpha) = \mathfrak{f}(\mu) \ominus \alpha$,
(P3) $\mathfrak{f}(0) = 0$,

where on $[0,\infty]^X$ the order, addition and $\ominus$ are defined pointwise.

From (P2) and (P3) it follows that for any α constant also $\mathfrak{f}(\alpha) \leqslant \alpha$. Applying (P1) it follows that with $\mu \leqslant \alpha$ also $\mathfrak{f}(\mu) \leqslant \alpha$, whenever $\mathfrak{f} \in \mathsf{P}(X)$.

We use the following notations. For $x \in X$ we let

$$\mathrm{ev}_x : BX \to [0,\infty] : \mu \mapsto \mu(x)$$

and for $\mu \in BX$ we let

$$\mathrm{ev}_\mu : \mathsf{P}X \to [0,\infty] : \mathfrak{f} \mapsto \mathfrak{f}(\mu)$$

Note that in view of the previous observations, both functions are bounded.

We define the following triple (P, e, m),

$$\mathsf{P} : \mathsf{Set} \to \mathsf{Set} \begin{cases} X \mapsto \mathsf{P}X \\ f : X \to Y \mapsto [\mathsf{P}X \to \mathsf{P}Y : \mathfrak{f} \mapsto \mathfrak{f}(- \cdot f)] \end{cases}$$

Further, for any set X let

$$e_X : X \to \mathsf{P}X : x \mapsto \mathrm{ev}_x$$

and

$$m_X : \mathsf{P}^2 X \to \mathsf{P}X : F \mapsto \big[BX \to [0,\infty] : \mu \mapsto F(\mathrm{ev}_\mu)\big]$$

2.1. Theorem. *$\mathbb{P} := (\mathsf{P}, e, m)$ is a monad over* Set.

Proof. That P is a well-defined functor and that for any $x \in X$ and $F \in \mathsf{P}^2X$ both $e_X(x)$ and $m_X(F)$ satisfy (P1)–(P3) is easily verified and we leave this to the reader. To see that the diagrams

$$\begin{tikzcd} \mathsf{P}X \arrow[r, "\mathsf{P}e_X"] \arrow[dr, "1_{\mathsf{P}X}"'] & \mathsf{P}^2X \arrow[d, "m_X"] \\ & \mathsf{P}X \end{tikzcd} \qquad \begin{tikzcd} \mathsf{P}X \arrow[r, "e_{\mathsf{P}X}"] \arrow[dr, "1_{\mathsf{P}X}"'] & \mathsf{P}^2X \arrow[d, "m_X"] \\ & \mathsf{P}X \end{tikzcd}$$

commute, let $\mathfrak{f} \in \mathsf{P}X$ and $\mu \in BX$, then we have

$$m_X\big(\mathfrak{f}(- \cdot e_X)\big)(\mu) = \mathfrak{f}(\mathrm{ev}_\mu \cdot e_X) = \mathfrak{f}(\mu)$$

and

$$m_X(\mathrm{ev}_\mathfrak{f})(\mu) = \mathrm{ev}_\mu(\mathfrak{f}) = \mathfrak{f}(\mu)$$

To see that the diagram

$$\begin{tikzcd} \mathsf{P}^3X \arrow[r, "\mathsf{P}m_X"] \arrow[d, "m_{\mathsf{P}X}"'] & \mathsf{P}^2X \arrow[d, "m_X"] \\ \mathsf{P}^2X \arrow[r, "m_X"'] & \mathsf{P}X \end{tikzcd}$$

commutes, first note that if $F \in \mathsf{P}^2X$ and $\mu \in BX$, then we have $\mathrm{ev}_{\mathrm{ev}_\mu}(F) = m_X F(\mu) = \mathrm{ev}_\mu \cdot m_X(F)$. Hence it follows that for $\boldsymbol{F} \in \mathsf{P}^3X$:

$$\begin{aligned} m_X\big(m_{\mathsf{P}X}(\boldsymbol{F})\big)(\mu) &= m_{\mathsf{P}X}(\boldsymbol{F})(\mathrm{ev}_\mu) \\ &= \boldsymbol{F}(\mathrm{ev}_{\mathrm{ev}_\mu}) \\ &= \boldsymbol{F}(\mathrm{ev}_\mu \cdot m_X) \\ &= m_X\big(\boldsymbol{F}(- \cdot m_X)\big)(\mu) \\ &= m_X\big(\mathsf{P}m_X(\boldsymbol{F})\big)(\mu) \qquad \square \end{aligned}$$

We call the monad $\mathbb{P} = (\mathsf{P}, e, m)$ the *functional power monad*. Our choice of the setting and the conditions (P1)–(P3) for this monad has been dictated by the purpose of the present paper. It is possible to weaken the axioms considerably to obtain a more general and equally interesting monad. Related ideas in this direction in a more general setting can be found in [6] and [13].

The order on $\mathsf{P}X$ is defined pointwise. Moreover there are some operations which we have to be able to perform on functions in $\mathsf{P}X$, notably a translation. Given $\mathfrak{f} \in \mathsf{P}X$ and $\alpha \in [0, \infty]$, and again using the same notation for the corresponding constant function on BX, for any $\mu \in BX$ we define $\mathfrak{f} \ominus \alpha(\mu) := \mathfrak{f}(\mu) \ominus \alpha$.

2.2. Proposition. *If $\mathfrak{f} \in \mathsf{P}X$ then for any α also $\mathfrak{f} \ominus \alpha \in \mathsf{P}X$.*

Proof. By straightforward verification. $\square$

Finally we will also require what we call the *characteristic value* of $\mathfrak{f} \in \mathsf{P}X$,

$$c(\mathfrak{f}) := \sup\big\{\alpha \in [0, \infty] \;\big|\; \mathfrak{f}(\alpha) = 0\big\}$$

2.3. Proposition. *For any $\mathfrak{f} \in \mathsf{P}X$ and $\alpha \in [0, \infty]$: $\mathfrak{f}(\alpha + c(\mathfrak{f})) = \alpha$.*

Proof. From (P2) it immediately follows that $\mathfrak{f}(\alpha + c(\mathfrak{f})) \leqslant \alpha$. Suppose that $\mathfrak{f}(\alpha + c(\mathfrak{f})) =: \beta < \alpha$, then we would have $\mathfrak{f}(\alpha + c(\mathfrak{f}) \ominus \beta) = 0$ which is impossible by definition of $c(\mathfrak{f})$. $\square$

We call the element $\mathfrak{f}$ which is identically 0 the improper element in $\mathsf{P}X$.

3. Submonads of $\mathbb{P}$

Given two monads $\mathbb{S} = (S, d, n)$ and $\mathbb{T} = (T, e, m)$ over Set recall that a *monad morphism* from $\mathbb{S}$ to $\mathbb{T}$ is a natural transformation $\alpha : S \Rightarrow T$ such that for any set X the diagrams

$$\begin{tikzcd} X \arrow[r, "d_X"] \arrow[dr, "e_X"'] & SX \arrow[d, "\alpha_X"] \\ & TX \end{tikzcd} \qquad \begin{tikzcd} S^2X \arrow[r, "n_X"] \arrow[d, "\alpha^2_X"'] & SX \arrow[d, "\alpha_X"] \\ T^2X \arrow[r, "m_X"'] & TX \end{tikzcd}$$

commute. In the case that for any set X the map α_X is a monomorphism, we will speak of a *submonad*, in particular in this paper these maps will often be canonical subset injections. In this section we consider some extra conditions defining corresponding subsets of $\mathsf{P}(X)$.

3.1. Definition. For any set X consider the following extra condition on the functions $\mathfrak{f} \in \mathsf{P}(X)$

(π) $\mathfrak{f}$ is a lattice morphism i.e. $\mathfrak{f}$ preserves binary infs and sups.

Let $a^{\mathsf{P}}_X : \mathsf{P}_\pi X \to \mathsf{P}X$ be the canonical subset injection, where $\mathsf{P}_\pi X$ consists of those $\mathfrak{f}$ satisfying (π).

Observe that for any $f : X \to Y$ we have that $\mathsf{P}_\pi(f) : \mathsf{P}_\pi X \to \mathsf{P}_\pi Y$ is well defined as being the restriction of $\mathsf{P}(f)$ to $\mathsf{P}_\pi X$. Moreover for any set X we have $\mathrm{ev}_x \in \mathsf{P}_\pi X$ for every $x \in X$. With ev^π_μ the restriction of ev_μ to $\mathsf{P}_\pi X$ we have

3.2. Proposition. $\mathbb{P}_\pi = (\mathsf{P}_\pi, e^\pi, m^\pi)$ *with*

$$e^\pi_X : X \to \mathsf{P}_\pi X : x \mapsto \mathrm{ev}_x$$

and

$$m^\pi_X : \mathsf{P}^2_\pi X \to \mathsf{P}_\pi X : F \mapsto \big[BX \to [0,\infty] : \mu \mapsto F\big(\mathrm{ev}^\pi_\mu\big)\big]$$

is a submonad of $\mathbb{P}$.

Proof. It is clear that the identities needed for the natural transformation $a^{\mathsf{P}} : \mathbb{P}_\pi \Rightarrow \mathbb{P}$ to be a monad morphism are satisfied. $\square$

In order to give an alternative description of $\mathbb{P}_\pi$, recall the basics on so-called *prime functional ideals* [11,10]. A *functional ideal* on X is an ideal $\mathfrak{F}$ in BX which is *saturated* in the sense that for all $\mu \in BX$:

$$\forall \epsilon > 0\ \exists \varphi \in \mathfrak{F}\colon\quad \mu \leqslant \varphi + \epsilon \quad \Rightarrow \quad \mu \in \mathfrak{F}$$

If a collection $\mathfrak{B}$ of bounded functions is an ideal then we can *saturate* it by putting $\mathfrak{B}^\sigma := \{\mu \mid \forall \epsilon > 0\ \exists \varphi \in \mathfrak{B}\colon\ \mu \leqslant \varphi + \epsilon\}$, which then is a functional ideal. Given a functional ideal $\mathfrak{F}$ its *characteristic value* is

$$c(\mathfrak{F}) := \sup_{\mu \in \mathfrak{F}} \inf_{x \in X} \mu(x) = \sup\{\alpha \mid \alpha \text{ constant},\ \alpha \in \mathfrak{F}\}$$

There is only one functional ideal on X which has an infinite characteristic value, namely the set BX consisting of all bounded functions. To make clear when we interpret this as a functional ideal we denote it as $\mathfrak{Z}_X$. A functional ideal with a finite characteristic value is called a *proper functional ideal* and $\mathfrak{Z}_X$ is called the *improper functional ideal*.

Further, if $\mathfrak{F}$ is a functional ideal and $\alpha < \infty$ then we define

$$\mathfrak{F} \oplus \alpha := \{\mu \mid \exists \psi \in \mathfrak{F}\colon \mu \leqslant \psi + \alpha\}$$

which is again a functional ideal. The collection of functional ideals on X is ordered by putting

$$\mathfrak{F} \leqslant \mathfrak{G} \quad \Leftrightarrow \quad \mathfrak{F} \subseteq \mathfrak{G}$$

and it is a complete lattice. Arbitrary infima always exist and are proper as long as at least one of the functional ideals involved is proper. Arbitrary suprema always exist, but in general the supremum of proper functional ideals need no longer be proper.

A functional ideal $\mathfrak{U}$ is called *prime* if for all bounded functions μ and ν,

$$\mu \wedge \nu \in \mathfrak{U} \quad \Rightarrow \quad \mu \in \mathfrak{U} \text{ or } \nu \in \mathfrak{U}$$

Note that obviously $\mathfrak{Z}_X$ is a prime functional ideal. Moreover, it is the only maximal prime functional ideal on X. We denote by πX the set of *all prime functional ideals*. The following defines a functor.

$$\pi : \mathsf{Set} \to \mathsf{Set} : \begin{cases} X \mapsto \pi X \\ f \mapsto [\pi f : \pi X \to \pi Y : \mathfrak{F} \mapsto \{\mu \in B(Y) \mid \mu \circ f \in \mathfrak{F}\}] \end{cases}$$

From [10] we recall the relation between ultrafilters and prime functional ideals. On a set X let

$$\beta X = \{\mathcal{U} \mid \text{ultrafilter on } X\} \cup \mathcal{Z}_X$$

with $\mathcal{Z}_X = 2^X$ the improper filter. Note that usually βX stands for the set of all (proper) ultrafilters, but that in this paper the improper filter is included, which is nicer with respect to the description of the following functions

$$\omega_X : \beta X \to \pi X : \mathcal{U} \mapsto \{\theta_F \wedge \omega \mid F \in \mathcal{U},\ \omega < \infty\}^{\sigma}$$

where for any set $A \subseteq X$ we put θ_A the function which is zero on A and infinite on $X \setminus A$. Note that $\omega_X(\mathcal{Z}_X) = \mathfrak{Z}_X$. For any $\mathcal{U} \in \beta X$ an equivalent description of $\omega_X(\mathcal{U})$ is known to be given by

$$\omega_X(\mathcal{U}) = \{\mu \text{ bounded} \mid Z(\mu) \in \mathcal{U}\}^{\sigma}$$

where as before $Z(\mu)$ is the zeroset of μ. In particular for $x \in X$ we get $\omega_X(\dot{x}) = \{\nu \in BX \mid \nu(x) = 0\}$ and more generally for $\alpha < \infty$,

$$\omega_X(\dot{x}) \oplus \alpha = \{\nu \in BX \mid \nu(x) \leqslant \alpha\}$$

Clearly $\omega_X(\dot{x}) \oplus \alpha$ is a proper prime functional ideal. The other way around with a prime functional ideal we can associate an element of βX by putting

$$\iota_X : \pi X \to \beta X : \mathfrak{F} \mapsto \begin{cases} \{\{\mu < \beta\} \mid \mu \in \mathfrak{F},\ c(\mathfrak{F}) < \beta\} & \mathfrak{F} \text{ proper} \\ \mathcal{Z}_X & \mathfrak{F} = \mathfrak{Z}_X \end{cases}$$

Furthermore in the sequel we require the following functions.

$$l : BX \to B\pi X : \nu \mapsto [\mathfrak{F} \mapsto \inf\{\alpha \mid \nu \in \mathfrak{F} \oplus \alpha\}]$$

3.3. Proposition. *([11]) For any $\mathfrak{F}$ the function $l(-)(\mathfrak{F})$ is a lattice morphism and moreover*

(1) *If α is constant then $l(\alpha)(\mathfrak{F}) = \alpha \ominus c(\mathfrak{F})$ in particular $l(\alpha) \leqslant \alpha$.*
(2) *If α is constant then for any $\nu \in BX$: $l(\nu \ominus \alpha)(-) = l(\nu)(-) \ominus \alpha$.*
(3) *$l(\nu)$ is an extension of μ in the sense that $l(\nu)(\omega_X(\dot{x})) = \nu(x)$ for any $x \in X$.*
(4) *If $f : X \to Y$ and $\nu \in BY$ then $l(\nu \circ f)(-) = l(\nu)(-) \circ \pi f(-)$.*
(5) *For any ν and $\mathfrak{F}$, $l(\nu)(\mathfrak{F}) = 0$ if and only if $\nu \in \mathfrak{F}$.*

For any set X we put

$$e_X : X \to \pi X : x \mapsto \omega_X(\dot{x})$$

and

$$m_X : \pi^2 X \to \pi X : \Phi \mapsto \{\nu \mid l(\nu)(-) \in \Phi\}$$

Note that $m_X(\mathfrak{Z}_{\pi X}) = \mathfrak{Z}_X$ and also $m_X(\omega_{\pi X}(\dot{\mathfrak{Z}}_X) \oplus \alpha) = \mathfrak{Z}_X$ for any $\alpha < \infty$. In [11] it was seen that the triple $\boldsymbol{\pi} := (\pi, e, m)$ is a monad on Set which we call the *prime functional ideal monad* and that the following is an alternative description for the multiplication

$$m_X(\Phi) = \bigvee_{\mathcal{A} \in \iota_{\pi X}(\Phi)} \bigcap_{\mathfrak{U} \in \mathcal{A}} \mathfrak{U} \oplus c(\Phi)$$

and

$$c(m_X(\Phi)) = c(\Phi) + \inf_{\mathcal{A} \in \iota_{\pi X}(\Phi)} \sup_{\mathfrak{G} \in \mathcal{A}} c(\mathfrak{G})$$

For any set X we now consider the map

$$a_X : \pi X \to \mathsf{P}X : \mathfrak{F} \mapsto l(-)(\mathfrak{F})$$

It is easily seen and follows from 3.3 that this map is always well defined. We will now go on to prove that the collection of maps $(a_X)_X$ actually defines a natural transformation a between the set functors π and P which will be the crucial link between the various monads we will encounter.

In what follows, unless absolutely necessary to make a distinction, we will denote unit and multiplication in the various monads which we encounter by the same symbols e and m respectively. This will not lead to confusion as it will always be clear from the context what is meant.

3.4. Proposition. *$a : \pi \Rightarrow \mathsf{P}$ is a natural transformation, with each a_X mono and such that the diagrams*

$$\begin{tikzcd} X \arrow[r, "e_X"] \arrow[dr, "e_X"'] & \pi X \arrow[d, "a_X"] \\ & \mathsf{P}X \end{tikzcd}$$

and

$$\begin{tikzcd} \pi^2 X \arrow[r, "m_X"] \arrow[d, "a_X^2"'] & \pi X \arrow[d, "a_X"] \\ \mathsf{P}^2 X \arrow[r, "m_X"'] & \mathsf{P}X \end{tikzcd}$$

commute. Moreover $\boldsymbol{\pi} = (\pi, e, m)$ is a monad and $a : \boldsymbol{\pi} \to \mathbb{P}$ is a monad monomorphism.

Proof. First note that a is a natural transformation since for any $f : X \to Y$, $\mathfrak{F} \in \pi X$ and $\mu \in BY$: $\mu \cdot f \in \mathfrak{F}$ if and only if $\mu \in \pi f(\mathfrak{F})$.

The first diagram commutes since for any $x \in X$ and $\mu \in BX$ we have $a_X \cdot e_X(x)(\mu) = l(\mu)(\omega_X(\dot{x})) = \mathrm{ev}_\mu(x)$. In order to prove that the second diagram commutes let $\Phi \in \pi^2 X$ then $a_X \cdot m_X(\Phi) = l(-)(m_X(\Phi))$ and $m_X \cdot a_X^2(\Phi) = l(\mathrm{ev}_-)(\pi a_X(\Phi))$. Let $\mu \in BX$ then

$$l(\mu)\big(m_X(\Phi)\big) = \inf\big\{\alpha \bigm| l\big((\mu) \ominus \alpha\big) \in \Phi\big\}$$

and

$$l(\mathrm{ev}_\mu)\big(\pi a_X(\Phi)\big) = \inf\big\{\alpha \bigm| \big((\mathrm{ev}_\mu) \ominus \alpha\big) \cdot a_X \in \Phi\big\}$$

and the result follows from 3.3 and the fact that for any $\mathfrak{F} \in \pi X$,

$$\big((\mathrm{ev}_\mu) \ominus \alpha\big) \cdot \alpha_X(\mathfrak{F}) = l(\mu)(\mathfrak{F}) \ominus \alpha$$

To see that, for any X, a_X is injective, note that if $\mathfrak{F}$ and $\mathfrak{G}$ are in πX and $\mu \in \mathfrak{F} \setminus \mathfrak{G}$ then it follows from 3.3(5) that $l(\mu)(\mathfrak{F}) = 0$ whereas $l(\mu)(\mathfrak{G}) \neq 0$.

It follows that $\boldsymbol{\pi}$ is a monad and a is a monad morphism. $\square$

Remark that the final conclusion in the previous proposition, that $\boldsymbol{\pi} = (\pi, e, m)$ is a monad, was already obtained in [11]. We will next show that this monad in fact is isomorphic to $\mathbb{P}_\pi$. The associated maps $a_X : \pi X \to \mathsf{P}X$ clearly take values in $\mathsf{P}_\pi X$. We now show that the image of a_X is the whole of $\mathsf{P}_\pi X$. For any $\mathfrak{f} \in \mathsf{P}X$ let $Z(\mathfrak{f})$ stand for the zeroset of $\mathfrak{f}$, i.e. $Z(\mathfrak{f}) := \{\mu \in BX \mid \mathfrak{f}(\mu) = 0\}$.

3.5. Theorem. *For any set X there is a one-to-one correspondence between prime functional ideals on X and functions $\mathfrak{f} \in \mathsf{P}_\pi X$.*

This correspondence is given by

$$\pi X \to \mathsf{P}_\pi X : \mathfrak{F} \mapsto l(-)(\mathfrak{F}) \quad \textit{and} \quad \mathsf{P}_\pi X \to \pi X : \mathfrak{f} \mapsto Z(\mathfrak{f})$$

Moreover $l(-)(Z(\mathfrak{f})) = \mathfrak{f}$ and $Z(l(-)(\mathfrak{F})) = \mathfrak{F}$.

Proof. First we need to show that for any $\mathfrak{f} \in \mathsf{P}X$ which satisfies the given properties $Z(\mathfrak{f})$ is indeed a prime functional ideal. The ideal properties follow from the fact that $\mathfrak{f}$ is order- and sup-preserving, and the prime property follows from the fact that $\mathfrak{f}$ is inf-preserving. To prove the saturation property let $\mu \in BX$ be such that for all $\epsilon > 0$ there exists a $\mu_\epsilon \in Z(\mathfrak{f})$ with $\mu \leqslant \mu_\epsilon + \epsilon$ then it follows from (P2) that

$$0 = \mathfrak{f}(\mu_\epsilon) = \mathfrak{f}(\mu_\epsilon + \epsilon) \ominus \epsilon$$

so that $\mathfrak{f}(\mu_\epsilon \oplus \epsilon) \leqslant \epsilon$ and hence by (P1) and the arbitrariness of ϵ it follows that $\mathfrak{f}(\mu) = 0$.

Conversely, that for any $\mathfrak{F} \in \pi X$, $l(-)(\mathfrak{F})$ is a lattice morphism was shown in 3.3. Finally, given $\mathfrak{f} \in \mathsf{P}_\pi X$ and making use of the appropriate properties, we have

$$\begin{aligned} l(\mu)\big(Z(\mathfrak{f})\big) &= \inf\big\{\alpha \;\big|\; \mu \in Z(\mathfrak{f}) \oplus \alpha\big\} \\ &= \inf\big\{\alpha \;\big|\; \mathfrak{f}(\mu \ominus \alpha) = 0\big\} \\ &= \inf\big\{\alpha \;\big|\; \mathfrak{f}(\mu) \leqslant \alpha\big\} \\ &= \mathfrak{f}(\mu) \end{aligned}$$

and given $\mathfrak{F} \in \pi X$ it follows from the saturation property of functional ideals that

$$Z\big(l(-)(\mathfrak{F})\big) = \big\{\mu \;\big|\; l(\mu)(\mathfrak{F}) = 0\big\} = \{\mu \mid \mu \in \mathfrak{F}\} = \mathfrak{F} \qquad \square$$

3.6. Corollary. *The monads* $\boldsymbol{\pi} = (\pi, e, m)$ *and* $\mathbb{P}_\pi := (\mathsf{P}_\pi, e^\pi, m^\pi)$ *are isomorphic.*

The isomorphism $\boldsymbol{\pi} \Rightarrow \mathbb{P}_\pi$ is again denoted by a. In the foregoing we have been concerned with the properties of $l(-)(-)$ in the first variable, but as far as the relation between the monads $\boldsymbol{\pi}$ and $\mathbb{P}_\pi$ is concerned it are the properties in the second variable which are important. For any X the order-reversing isomorphism $a_X : \pi X \to \mathsf{P}_\pi X$, is structure-preserving.

3.7. Theorem. *For any* X, $\mathsf{P}_\pi X$ *is a complete lattice equipped with a family of translations which preserve the lattice structure.*

(1) *For any family* $(\mathfrak{f}_j)_j$ *in* $\mathsf{P}_\pi X$:

$$\Big(\bigvee_j \mathfrak{f}_j\Big)(-) = \sup_j\big(\mathfrak{f}_j(-)\big) \text{ and } \Big(\bigwedge_j \mathfrak{f}_j\Big)(-) = l(-)\Big(\bigvee_j Z(\mathfrak{f}_j)\Big)$$

(2) *As far as the relationship between* πX *and* $\mathsf{P}_\pi X$ *is concerned, for any families* $(\mathfrak{f}_j)_j$ *in* $\mathsf{P}_\pi X$, $(\mathfrak{F}_j)_j$ *in* πX *and for any* $\mathfrak{f} \in \mathsf{P}_\pi X$, $\mathfrak{F} \in \pi X$ *and* $\gamma \in [0, \infty]$:

$$\begin{array}{ccc} \pi X & \xrightarrow{\quad l(-)(-) \quad} & \mathsf{P}_\pi X \\ \bigcap_j \mathfrak{F}_j & \longrightarrow & \bigvee_j l(-)(\mathfrak{F}_j) \\ \bigvee_j \mathfrak{F}_j & \longrightarrow & \bigwedge_j l(-)(\mathfrak{F}_j) \\ \mathfrak{F} \oplus \gamma & \longrightarrow & l(-)(\mathfrak{F}) \ominus \gamma \end{array} \qquad \begin{array}{ccc} \mathsf{P}_\pi X & \xrightarrow{\quad Z \quad} & \pi X \\ \bigvee_j \mathfrak{f}_j & \longrightarrow & \bigcap_j Z(\mathfrak{f}_j) \\ \bigwedge_j \mathfrak{f}_j & \longrightarrow & \bigvee_j Z(\mathfrak{f}_j) \\ \mathfrak{f} \ominus \gamma & \longrightarrow & Z(\mathfrak{f}) \oplus \gamma \end{array}$$

Proof. We only treat the case of the infimum in (1), leaving the remaining verifications to the reader. First note that given the family $(\mathfrak{f}_j)_j$ in $\mathsf{P}_\pi X$, for all j, $l(-)(\vee_j Z(\mathfrak{f}_j)) \leqslant \mathfrak{f}_j$ from 3.3. If $\mathfrak{f} \leqslant \mathfrak{f}_j$ for all j, then, since $\mathfrak{f} = l(-)(Z(\mathfrak{f}))$ and $\bigvee_j Z(\mathfrak{f}_j) \subseteq Z(\mathfrak{f})$ we have $\mathfrak{f} \leqslant l(-)(\bigvee_j Z(\mathfrak{f}_j))$. $\square$

Note that the supremum in the lattice $\mathsf{P}_\pi X$ is given by pointwise supremum but that the infimum is quite different. We always have that $\bigwedge_j \mathfrak{f}_j \leqslant \inf_j \mathfrak{f}_j$ but even for an infimum of two functions this inequality is in general strict, e.g. $\mathrm{ev}_x \wedge \mathrm{ev}_y = 0$ if $x \neq y$.

Next we consider some further condition defining another subset of $\mathsf{P}(X)$.

3.8. Definition. For any set X consider the following extra conditions on the functions $\mathfrak{f} \in \mathsf{P}(X)$:

(0) $\mathfrak{f}$ is non-zero,
(π) $\mathfrak{f}$ is a lattice morphism,
(D) $\mathfrak{f}$ is a sup-map (which here means that it preserves bounded suprema).

Let $d_X^{\mathsf{P}} : \mathsf{P}_{\mathsf{D}} X \to \mathsf{P}X$ be the canonical subset injection, where $\mathsf{P}_{\mathsf{D}} X$ consists of those $\mathfrak{f}$ satisfying all of the foregoing conditions.

Observe that for any $f : X \to Y$ we have that $\mathsf{P}_{\mathsf{D}}(f) : \mathsf{P}_{\mathsf{D}} X \to \mathsf{P}_{\mathsf{D}} Y$ is well defined as being the restriction of $\mathsf{P}(f)$ to $\mathsf{P}_{\mathsf{D}} X$. Moreover for any set X we have $\mathrm{ev}_x \in \mathsf{P}_{\mathsf{D}} X$ for every $x \in X$. With $\mathrm{ev}_\mu^{\mathsf{D}}$ the restriction of ev_μ to $\mathsf{P}_{\mathsf{D}} X$ we have:

3.9. Proposition. $\mathbb{P}_{\mathsf{D}} = (\mathsf{P}_{\mathsf{D}}, e^{\mathsf{D}}, m^{\mathsf{D}})$ *with*

$$e^{\mathsf{D}}_X : X \to \mathsf{P}_{\mathsf{D}}X : x \mapsto \mathrm{ev}_x$$

and

$$m^{\mathsf{D}}_X : \mathsf{P}^2_{\mathsf{D}}X \to \mathsf{P}_{\mathsf{D}}X : F \mapsto \big[BX \to [0,\infty] : \mu \mapsto F\big(\mathrm{ev}^{\mathsf{D}}_\mu\big)\big]$$

is a submonad of $\mathbb{P}$.

Proof. It is clear that the identities needed for the natural transformation $d^{\mathsf{P}} : \mathbb{P}_{\mathsf{D}} \Rightarrow \mathbb{P}$ to be a monad morphism are satisfied. □

In order to give an alternative description of $\mathbb{P}_{\mathsf{D}}$ for any set X we consider the following subset of πX,

$$\mathsf{D}X := \big\{\omega_X(\dot{x}) \oplus \alpha \;\big|\; x \in X,\ \alpha < \infty\big\}$$

and $\delta_X : \mathsf{D}X \to \pi X$ the canonical subset injection. The elements in $\mathsf{D}X$ can also be characterized as the proper prime functional ideals which are closed under the taking of bounded suprema.

The following defines a functor.

$$\mathsf{D} : \mathsf{Set} \to \mathsf{Set} : \begin{cases} X \mapsto \mathsf{D}X \\ f \mapsto [\mathsf{D}f : \mathsf{D}X \to \mathsf{D}Y : \omega_X(\dot{x}) \oplus \alpha \mapsto \omega_Y(f(\dot{x})) \oplus \alpha] \end{cases}$$

For any set X we put

$$e_X : X \to \mathsf{D}X : x \mapsto \omega_X(\dot{x})$$

and

$$m_X : \mathsf{D}^2X \to \mathsf{D}X : \Phi \mapsto \omega_X(\dot{x}) \oplus (\alpha + \beta)$$

where $\Phi = \omega_{\mathsf{D}X}(\dot{\mathfrak{F}}) \oplus \alpha$ and $\mathfrak{F} = \omega_X(\dot{x}) \oplus \beta$.

For any set X we now consider the map

$$d_X : \mathsf{D}X \to \mathsf{P}X : \omega_X(\dot{x}) \oplus \alpha \mapsto l(-)\big(\omega_X(\dot{x}) \oplus \alpha\big)$$

where $l(\nu)(\omega_X(\dot{x}) \oplus \alpha) = \nu(x) \ominus \alpha$. It is easily seen that this map is the restriction of a_X to $\mathsf{D}X$. We will now go on to prove that the collection of maps $(d_X)_X$ actually defines a natural transformation d between the set functors D and P.

3.10. Proposition. $d : \mathsf{D} \Rightarrow \mathsf{P}$ *is a natural transformation, with each* d_X *mono and such that the diagrams*

$$\begin{tikzcd} X \arrow[r, "e_X"] \arrow[dr, "e_X"'] & \mathsf{D}X \arrow[d, "d_X"] \\ & \mathsf{P}X \end{tikzcd}$$

and

$$\begin{tikzcd} \mathsf{D}^2X \arrow[r, "m_X"] \arrow[d, "d^2_X"'] & \mathsf{D}X \arrow[d, "d_X"] \\ \mathsf{P}^2X \arrow[r, "m_X"'] & \mathsf{P}X \end{tikzcd}$$

commute. Moreover $\mathbb{D} = (\mathsf{D}, e, m)$ *is a monad and* $d : \mathbb{D} \to \mathbb{P}$ *is a monad monomorphism.*

Proof. First note that d is a natural transformation since for any $f : X \to Y$, and $\omega_X(\dot{x}) \oplus \alpha \in \mathsf{D}X$ we have

$$\mathsf{P}f\big(l(-)\big(\omega_X(\dot{x}) \oplus \alpha\big)\big) = l(-)\big(\omega_Y\big(f(\dot{x})\big) \oplus \alpha\big)$$

The first diagram commutes since clearly d_X is the restriction of a_X to $\mathsf{D}X$ and so we can apply 3.10.

In order to prove that the second diagram commutes let $\Phi \in \mathsf{D}^2X$, $\Phi = \omega_{\mathsf{D}X}(\dot{\mathfrak{F}}) \oplus \alpha$ with $\mathfrak{F} = \omega_X(\dot{x}) \oplus \beta$. Then clearly $\Phi \in \pi^2X$ and so the result follows again from 3.10 and the fact that $m_X : \mathsf{D}^2X \to \mathsf{D}X$ is a restriction of $m_X : \pi^2X \to \pi X$, $d_X = a_X|_{\mathsf{D}X}$ and $d^2_X = a^2_X|_{\mathsf{D}^2X}$. These observations also imply that for any X, d_X is injective. It follows that $\mathbb{D}$ is a monad and d is a monad morphism. □

The associated maps $d_X : \mathsf{D}X \to \mathsf{P}X$ are not onto and we now characterize the image of d_X.

3.11. Theorem. *The one-to-one correspondence between πX and $\mathsf{P}_\pi X$ restricts to a one-to-one correspondence between $\mathsf{D}X$ and non-zero functions $\mathfrak{f} \in \mathsf{P}_\pi X$ which preserve bounded suprema, i.e. there is a one-to-one correspondence between principal prime functional ideals on X and the functions in $\mathsf{P}_{\mathsf{D}} X$.*

Proof. We already know what the canonical bijection between πX and $\mathsf{P}_\pi X$ is from 3.5. On the one hand suppose that $(\mu_j)_j$ is a family in BX such that their supremum is bounded, then for any $x \in X$ and any $\alpha < \infty$,

$$\begin{aligned} l\Bigl(\sup_j \mu_j\Bigr)\bigl(\omega_X(\dot{x}) \oplus \alpha\bigr) &= \inf\Bigl\{\beta \Bigm| \sup_j \mu_j \in \omega_X(\dot{x}) \oplus (\alpha+\beta)\Bigr\} \\ &= \inf\Bigl\{\beta \Bigm| \sup_j \mu_j(x) \leqslant \alpha+\beta\Bigr\} \\ &= \inf\Bigl(\bigcap_j \{\beta \mid \mu_j(x) \leqslant \alpha+\beta\}\Bigr) \\ &= \sup_j l(\mu_j)\bigl(\omega_X(\dot{x}) \oplus \alpha\bigr) \end{aligned}$$

On the other hand suppose that $\mathfrak{f} \in \mathsf{P}_\pi X$ preserves bounded suprema. Hence, if $(\mu_j)_j$ is a family with bounded supremum all the members of which are in $Z(\mathfrak{f})$ then also $\sup_j \mu_j \in Z(\mathfrak{f})$ which means that for any $\omega < \infty$, $\vee Z(\mathfrak{f}) \wedge \omega \in Z(\mathfrak{f})$, i.e. $Z(\mathfrak{f})$ is a principal prime functional ideal and hence of type $\omega_X(\dot{x}) \oplus \alpha$ for some x and α. $\square$

3.12. Theorem. *The monads $\mathbb{D}$ and $\mathbb{P}_{\mathsf{D}}$ are isomorphic.*

The isomorphism $\mathbb{D} \Rightarrow \mathbb{P}_{\mathsf{D}}$ is again denoted by d.
Next we consider the following extra conditions defining yet another corresponding subset of $\mathsf{P}(X)$.

3.13. Definition. For any set X consider the following extra conditions on the functions $\mathfrak{f} \in \mathsf{P}(X)$:

(π) $\mathfrak{f}$ is a lattice morphism,
(β) for any $\alpha \in [0,\infty] : \mathfrak{f}(\alpha) = \alpha$.

Let $b_X^{\mathsf{P}} : \mathsf{P}_\beta X \to \mathsf{P}X$ be the canonical subset injection, where $\mathsf{P}_\beta X$ consists of those $\mathfrak{f}$ satisfying (π) and (β).

Observe that for any $f : X \to Y$ we have that $\mathsf{P}_\beta(f) : \mathsf{P}_\beta X \to \mathsf{P}_\beta Y$ is well defined as being the restriction of $\mathsf{P}(f)$ to $\mathsf{P}_\beta X$. Moreover for any set X we have $\mathrm{ev}_x \in \mathsf{P}_\beta X$ for every $x \in X$. With ev_μ^β the restriction of ev_μ to $\mathsf{P}_\beta X$ we have

3.14. Proposition. $\mathbb{P}_\beta = (\mathsf{P}_\beta, e^\beta, m^\beta)$ *with*

$$e_X^\beta : X \to \mathsf{P}_\beta X : x \mapsto \mathrm{ev}_x$$

and

$$m_X^\beta : \mathsf{P}_\beta^2 X \to \mathsf{P}_\beta X : F \mapsto \bigl[BX \to [0,\infty] : \mu \mapsto F\bigl(\mathrm{ev}_\mu^\beta\bigr)\bigr]$$

is a submonad of $\mathbb{P}$.

Proof. It is clear that the identities needed for the natural transformation $b^{\mathsf{P}} : \mathbb{P}_\beta \Rightarrow \mathbb{P}$ to be a monad morphism are satisfied. $\square$

In order to give an alternative description of $\mathbb{P}_\beta$ as before

$$\beta X = \{\mathcal{U} \mid \text{ultrafilter on } X\} \cup \mathcal{Z}_X$$

with $\mathcal{Z}_X$ the improper filter on X. The following defines a functor.

$$\beta : \mathsf{Set} \to \mathsf{Set} : \begin{cases} X \mapsto \beta X \\ f \mapsto [\beta f : \beta X \to \beta Y : \mathcal{U} \mapsto \operatorname{stack} f(\mathcal{U});\ \mathcal{Z}_X \mapsto \mathcal{Z}_Y] \end{cases}$$

For any set X we put

$$e_X : X \to \beta X : x \mapsto \dot{x}$$

and

$$m_X : \beta^2 X \to \beta X : \Xi \mapsto \bigl\{A \bigm| \{\mathcal{U} \in \beta X \mid A \in \mathcal{U}\} \in \Xi\bigr\}$$

As is well known from [12,2] or [1] this defines (an extended version) of the ultrafilter monad $\boldsymbol{\beta} := (\beta, e, m)$. The multiplication has also another characterization coming from the work of Kowalsky [7]. Given any set J, an ultrafilter $\mathcal{J}$ on J and a selection of ultrafilters $(\mathcal{U}_j)_{j\in J}$,

$$\mathcal{D}\bigl(\mathcal{J}, (\mathcal{U}_j)_j\bigr) := \bigvee_{A\in\mathcal{J}} \bigcap_{j\in A} \mathcal{U}_j$$

With $J = \beta X$ and the identity selection this is exactly the multiplication.

In order to prove that $\boldsymbol{\beta}$ is a submonad of $\boldsymbol{\pi}$ we rely on the description of the natural transformation introduced in [10] which we recalled after Proposition 3.2. With $\omega = (\omega_X)_X$ and $\iota = (\iota_X)_X$ we have:

3.15. Theorem. *$\omega : \boldsymbol{\beta} \Rightarrow \boldsymbol{\pi}$ is a monad (mono)morphism and $\iota : \boldsymbol{\pi} \Rightarrow \boldsymbol{\beta}$ is a monad morphism and $\boldsymbol{\beta}$ can be considered to be a submonad of $\boldsymbol{\pi}$.*

Proof. We only look at the diagrams involving the multiplication.

$$\begin{tikzcd} \beta^2 X \arrow[r, "m_X"] \arrow[d, "\omega_X^2"'] & \beta X \arrow[d, "\omega_X"] \\ \pi^2 X \arrow[r, "m_X"'] & \pi X \end{tikzcd} \qquad \begin{tikzcd} \pi^2 X \arrow[r, "m_X"] \arrow[d, "\iota_X^2"'] & \pi X \arrow[d, "\iota_X"] \\ \beta^2 X \arrow[r, "m_X"'] & \beta X \end{tikzcd}$$

First, note that in both the diagrams concerning ω_X and the one involving ι_X the improper filter, respectively the improper functional ideal, is mapped to the improper filter by both m_X and ω_X^2 and to the improper functional ideal by both m_X and ι_X^2 respectively. Hence we can restrict our attention to proper filters and functional ideals. First let $\Theta \in \beta^2 X$ be proper, then

$$\omega_X^2(\Theta) = \bigl\{\theta_{\mathcal{B}} \wedge \omega \bigm| \mathcal{B} \subseteq \pi X\colon \{\mathcal{U} \mid \omega_X \mathcal{U} \in \mathcal{B}\} \in \Theta\bigr\}^{\sigma}$$

and it follows that

$$m_X\bigl(\omega_X^2(\Theta)\bigr) = \bigvee_{\mathcal{B}\subseteq\pi X, \{\mathcal{U}\mid\omega_X\mathcal{U}\in\mathcal{B}\}\in\Theta} \bigcap_{\mathfrak{U}\in\mathcal{B}} \mathfrak{U} = \omega_X\bigl(m_X(\Theta)\bigr)$$

Second let $\Phi \in \pi^2 X$ be proper, then

$$\iota_X^2(\Phi) = \bigl\{\bigl\{\iota_X(\mathfrak{G}) \bigm| \mathfrak{G} \in \mathcal{A}\bigr\} \bigm| \mathcal{A} \in \iota_{\pi X}(\Phi)\bigr\}$$

from which it follows that

$$\iota_X\bigl(m_X(\Phi)\bigr) = \bigcup_{\mathcal{A}\in\iota_X^2(\Phi)} \bigcap_{\mathcal{U}\in\mathcal{A}} \mathcal{U} = m_X\bigl(\iota_X^2(\Phi)\bigr) \qquad \Box$$

The pair of transformations ω and ι give rise to an adjunction.

3.16. Proposition. *For every X we have $\omega_X \dashv \iota_X$, thus determining a Galois connection between πX and βX.*

Proof. This follows from the facts established in [10] and [11] that for any $\mathcal{U} \in \beta X$,

$$\iota_X\bigl(\omega_X(\mathcal{U})\bigr) = \mathcal{U}$$

and that for any $\mathfrak{F} \in \pi X$,

$$\omega_X\bigl(\iota_X(\mathfrak{F})\bigr) \subseteq \omega_X\bigl(\iota_X(\mathfrak{F})\bigr) + c(\mathfrak{F}) = \mathfrak{F} \qquad \Box$$

Composing the natural transformations and writing $b = a \circ \omega$ we now immediately obtain the next result.

3.17. Proposition. *$b : \beta \Rightarrow \mathsf{P}$ is a natural transformation, with each b_X mono and such that the diagrams*

$$\begin{tikzcd} X \arrow[r, "e_X"] \arrow[dr, "e_X"'] & \beta X \arrow[d, "b_X"] \\ & \pi X \end{tikzcd}$$

and

$$\begin{tikzcd} \beta^2 X \arrow[r, "m_X"] \arrow[d, "b_X^2"'] & \beta X \arrow[d, "b_X"] \\ \pi^2 X \arrow[r, "m_X"'] & \pi X \end{tikzcd}$$

commute. $b : \beta \to \mathbb{P}$ is a monad monomorphism.

The associated maps $b_X : \beta X \to \mathsf{P}X$ are not onto and we now characterize the image of b_X.

3.18. Theorem. *The one-to-one correspondence between πX and $\mathsf{P}_\pi X$ restricts to a one-to-one correspondence between the proper ultrafilters in βX and non-zero functions $\mathfrak{f} \in \mathsf{P}_\beta X$ The improper filter corresponds to the function which is identically equal to zero.*

Proof. Again, from 3.5, it suffices to note that on the one hand for any α constant and any ultrafilter $\mathcal{U}$:

$$l(\alpha)\big(\omega_X(\mathcal{U})\big) = \inf\big\{\beta \bigm| \alpha \in \omega_X(\mathcal{U}) \oplus \beta\big\} = \alpha$$

and on the other hand if $\mathfrak{f} \in \mathsf{P}_\pi X$ is such that for all α, $\mathfrak{f}(\alpha) = \alpha$ then $Z(\mathfrak{f})$ is a proper prime functional ideal with only the zero function as constant member and hence $c(Z(\mathfrak{f})) = 0$. The conclusion follows from 3.16. $\square$

Finally we can also put the identity monad into the picture. Let $\mathbb{I} := (\mathsf{I}, e, m)$ stand for the identity monad and let

$$\kappa_X : \mathsf{I}X \to \pi X : x \mapsto \omega_X(\dot{x})$$

3.19. Theorem. *$\kappa : \mathbb{I} \Rightarrow \boldsymbol{\pi}$ is a monad (mono)morphism and $\mathbb{I}$ can be considered to be a submonad of $\boldsymbol{\pi}$.*

Proof. We leave this to the reader. $\square$

Since for any set X, $\mathsf{I}X = X$ can be canonically embedded in both βX and $\mathsf{D}X$ the characterization of the image of $\mathsf{I}X$ under $(a \cdot \kappa)_X$ is obtained by the combined properties of 3.11 and 3.18. However, this leads to a far more concise description.

3.20. Theorem. *The one-to-one correspondence between πX and $\mathsf{P}_\pi X$ restricts to a one-to-one correspondence between principal ultrafilters on X (or the points of X) and non-zero functions $\mathfrak{f} \in \mathsf{P}X$ fulfilling the following properties*:

(π) *$\mathfrak{f}$ is a lattice morphism.*
(D) *$\mathfrak{f}$ is a sup-map (which here means that it preserves bounded suprema).*
(β) *For any α constant*: $\mathfrak{f}(\alpha) = \alpha$.

The functions fulfilling these properties are exactly the evaluation functions ev_x, $x \in X$.

Proof. All we need to show is that functions fulfilling the given conditions are exactly the evaluation functions. That evaluation functions fulfil those conditions is trivial. Suppose that $\mathfrak{f}$ satisfies the conditions. Then by the results of 3.11 and 3.18, $\mathfrak{f} = l(-)(\omega_X(\dot{x}))$ for some $x \in X$ and it suffices to note that for any $\mu \in BX$,

$$\mathfrak{f}(\mu) = l(\mu)\big(\omega_X(\dot{x})\big) = \inf\big\{\alpha \bigm| \mu(x) \leqslant \alpha\big\} = \mu(x) = \mathrm{ev}_x(\mu) \qquad \square$$

For any given set X we will denote

$$\mathsf{P}_{\mathsf{I}} X := (a \cdot \kappa)_X(\mathsf{I}X)$$

The monad morphisms involving $\boldsymbol{\pi}$ translate to exceptionally simple monad morphisms involving $\mathbb{P}_\pi$. The monad morphisms $\delta', \kappa', \omega'$ are canonical subset injections and ι' is given by

$$\iota'_X(\mathfrak{f}) = \mathfrak{f}\big(- \oplus c(\mathfrak{f})\big)$$

This finally leads us to the following scheme of monads and monad morphisms with "pairwise" isomorphic monads in the first and second diamond.

$$\begin{tikzcd}
 & \mathbb{D} \arrow[rrr, "\approx"] \arrow[dr, "\delta"] & & & \mathbb{P}_{\mathbb{D}} \arrow[dr, "\delta'"] & & \\
\mathbb{I} \arrow[ur] \arrow[dr] \arrow[rr, "\kappa"] \arrow[rrr, bend left, "\approx"] & & \boldsymbol{\pi} \arrow[rrr, bend right, "\approx"] \arrow[dl, bend left, "\iota"] & \mathbb{P}_{\mathbb{I}} \arrow[ur] \arrow[rr, "\kappa'"] \arrow[dr] & & \mathbb{P}_{\boldsymbol{\pi}} \arrow[r] \arrow[dl, bend left, "\iota'"] & \mathbb{P} \\
 & \boldsymbol{\beta} \arrow[rrr, "\approx"] \arrow[ur, bend left, "\omega"] & & & \mathbb{P}_{\boldsymbol{\beta}} \arrow[ur, bend left, "\omega'"] & &
\end{tikzcd}$$

4. Lax extensions

We follow [15] and [14] for terminology on lax extensions of Set-functors and monads. Rel is the category of sets with relations as morphisms. Recall that in diagrams relations are denoted by a "strike-through" arrow.

4.1. Definitions. Given a monad $\mathbb{T} = (\mathsf{T}, e, m)$ over Set, by a *lax extension* $\bar{\mathsf{T}}$ of the functor T to Rel we mean the following. First $\bar{\mathsf{T}}$ is an endofunctor of Rel, satisfying:

(E1) For any $r : X \nrightarrow Y$ and $s : Y \nrightarrow Z$: $\bar{\mathsf{T}}s \cdot \bar{\mathsf{T}}r \leqslant \bar{\mathsf{T}}(s \cdot r)$.
(E2) For any $r : X \nrightarrow Y$ and $s : X \nrightarrow Y$: $s \leqslant r \Rightarrow \bar{\mathsf{T}}s \leqslant \bar{\mathsf{T}}r$.

And secondly it is an extension of T in the sense that:

(E3) For any set X: $\bar{\mathsf{T}}(X) = \mathsf{T}(X)$.
(E4) For any function f: $\mathsf{T}f \leqslant \bar{\mathsf{T}}f$ and $(\mathsf{T}f)^\circ \leqslant \bar{\mathsf{T}}f^\circ$.

Remark that $\bar{\mathsf{T}}$ is a lax functor, since by (E4) we also have $1_{T(X)} \leqslant \bar{\mathsf{T}}1_X$. In this paper all lax extensions of T also satisfy:

$\bar{\mathbb{T}}$ is a *lax extensions of the monad* $\mathbb{T}$ in the sense that the unit e and the multiplication m become op-lax transformations in the extension, i.e. for any $r : X \nrightarrow Y$ we have $e_Y \cdot r \leqslant \overline{T}r \cdot e_X$ and $m_Y \cdot \overline{T}^2 r \leqslant \overline{T}r \cdot m_X$.

Suppose $\mathbb{S} = (\mathsf{S}, d, n)$ is another monad. If $\bar{\mathsf{T}}$ and $\bar{\mathsf{S}}$ are lax extensions of the respective lax functors T and S, then $\alpha : \mathsf{T} \Rightarrow \mathsf{S}$ is said to extend to a *morphism of lax extensions* if $\alpha : \bar{\mathsf{T}} \Rightarrow \bar{\mathsf{S}}$ is an op-lax transformation in the sense that

$$\overline{S}r \leqslant \alpha_Y^\circ \cdot \overline{T}r \cdot \alpha_X$$

for any $r : X \nrightarrow Y$ [15]. It is known that given an extension $\bar{\mathsf{T}}$ the following defines a lax extension of S to Rel:

$$\bar{\mathsf{S}} : \mathsf{Rel} \to \mathsf{Rel} \begin{cases} \bar{\mathsf{S}}X := \mathsf{S}X & \text{for any set } X \\ \bar{\mathsf{S}}r := \alpha_Y^\circ \cdot \bar{\mathsf{T}}r \cdot \alpha_X & \text{for any } r : X \nrightarrow Y \end{cases}$$

$\bar{\mathsf{S}}$ is said to be the *initial extension* determined by $\bar{\mathsf{T}}$ and α [15]. The following extension condition which we introduce involves not only $\bar{\mathsf{T}}$ but also the initial extension $\bar{\mathsf{S}}$ derived from it. All initial extensions considered in this paper will satisfy this condition.

(IE) With $\rho := \alpha \circ \alpha$ for any set X and any relation $r : X \nrightarrow Y$,

$$\bar{\mathsf{S}}^2 r = \rho_Y^\circ \cdot \bar{\mathsf{T}}^2 r \cdot \rho_X$$

4.2. Proposition. *For an initial extension we have the following properties*:

(1) *If $\bar{\mathbb{T}}$ is a lax extension of the monad then d becomes op-lax.*
(2) *If $\bar{\mathbb{T}}$ is a lax extension of the monad and $\bar{\mathsf{T}}$ and $\bar{\mathsf{S}}$ satisfy* (IE) *then ρ and n become op-lax.*

Proof. (1) Suppose that $\bar{\mathbb{T}}$ is a lax extension of the monad. In the following diagrams the inequality to prove is denoted in brackets. That d becomes op-lax follows from noting that in the diagram

$$\begin{tikzcd}
X \arrow[r, "d_X"] \arrow[d, "r"', "\shortmid" marking] \arrow[dr, phantom, "(\leqslant)"] & \overline{\mathsf{S}}X \arrow[r, "\alpha_X"] \arrow[d, "\overline{\mathsf{S}}r"', "\shortmid" marking] \arrow[dr, phantom, "\leqslant"] & \overline{\mathsf{T}}X \arrow[d, "\overline{\mathsf{T}}r"', "\shortmid" marking] \\
Y \arrow[r, "d_Y"'] & \overline{\mathsf{S}}Y \arrow[r, "\alpha_Y"'] & \overline{\mathsf{T}}Y
\end{tikzcd}$$

the outer and right-hand squares $\leqslant$-commute.

(2) Suppose that $\overline{\mathbb{T}}$ is a lax extension of the monad and $\overline{\mathbb{T}}$ and $\overline{\mathbb{S}}$ satisfy (IE). It follows that

$$\begin{tikzcd}
\mathsf{S}^2X \arrow[r, "\rho_X"] \arrow[d, "\overline{\mathsf{S}}^2r"', "\shortmid" marking] \arrow[dr, phantom, "\leqslant"] & \mathsf{T}^2X \arrow[d, "\overline{\mathsf{T}}^2r", "\shortmid" marking] \\
\mathsf{S}^2Y \arrow[r, "\rho_Y"'] & \mathsf{T}^2Y
\end{tikzcd}$$

i.e. ρ is op-lax. That n becomes op-lax under the condition on $\overline{\mathsf{S}}^2$ follows from observing that in the diagram

$$\begin{tikzcd}
\overline{\mathsf{T}}^2X \arrow[rrr, "m_X"] \arrow[ddd, "\overline{\mathsf{T}}^2r"', "\shortmid" marking] & & & \overline{\mathsf{T}}X \arrow[ddd, "\overline{\mathsf{T}}r", "\shortmid" marking] \\
& \overline{\mathsf{S}}^2X \arrow[ul, "\rho_X"'] \arrow[r, "n_X"] \arrow[d, "\overline{\mathsf{S}}^2r"', "\shortmid" marking] \arrow[dr, phantom, "(\leqslant)"] & \overline{\mathsf{S}}X \arrow[ur, "\alpha_X"] \arrow[d, "\overline{\mathsf{S}}r", "\shortmid" marking] & \\
& \overline{\mathsf{S}}^2Y \arrow[dl, "\rho_Y"] \arrow[r, "n_Y"'] & \overline{\mathsf{S}}Y \arrow[dr, "\alpha_Y"] & \\
\overline{\mathsf{T}}^2Y \arrow[rrr, "m_Y"'] & & & \overline{\mathsf{T}}Y
\end{tikzcd}$$

(with $\geqslant$ in the left region and $\leqslant$ in the right region)

besides the indicated (lax)-commutations also the outer square $\leqslant$-commutes and hence the inner square $\leqslant$-commutes. Hence n is op-lax. □

5. Lax extension of the monad $\mathbb{P}$

We will now extend the monads introduced in the foregoing sections from Set to Rel. Since for all monads other than $\mathbb{P}$ this will be done via initial extensions, in a first step, it suffices to define the extension for $\mathbb{P}$ and hereto we need to define the action of this extension on relations. For this we require some properties which we will now first give. For $r : X \nrightarrow Y$ and any $\mu \in BY$ we define

$$r^\circ\mu(x) := \inf_{(x,y)\in r} \mu(y)$$

Note that if f is a function then $f^\circ\mu = \mu \cdot f$ and that in general $r^\circ\mu$ need not be bounded.

5.1. Lemma. *For relations $r : X \nrightarrow Y$, $s : X \nrightarrow Y$ and functions ν, ν_1 and ν_2 in BY the following properties hold*:

(1) $\nu_1 \leqslant \nu_2 \Rightarrow r^\circ\nu_1 \leqslant r^\circ\nu_2$,
(2) $r \subseteq s \Rightarrow s^\circ\nu \leqslant r^\circ\nu$,
(3) *for θ constant*: $r^\circ(\nu + \theta) = r^\circ\nu + \theta$,
(4) $r^\circ\nu_1 \vee r^\circ\nu_2 \leqslant r^\circ(\nu_1 \vee \nu_2)$.

Proof. By straightforward verification. □

5.2. Lemma. *For relations $r : X \nrightarrow Y$, $s : Y \nrightarrow Z$, $\omega < \infty$ and $\zeta \in BZ$ we obtain*

$$r^\circ\big(s^\circ\zeta \wedge \omega\big) \wedge \omega = (s \cdot r)^\circ\zeta \wedge \omega$$

Proof. Straightforward calculation. □

The following defines an extension of P to Rel. Given $r : X \nrightarrow Y$, $\mathfrak{f} \in \mathsf{P}X$ and $\mathfrak{g} \in \mathsf{P}Y$ we define

$$(\mathfrak{f}, \mathfrak{g}) \in \overline{\mathsf{P}}r \quad \text{if } \forall \mu \in BY,\ \forall \omega < \infty\text{:} \quad \mathfrak{f}\big(r^\circ\mu \wedge \omega\big) \leqslant \mathfrak{g}(\mu)$$

5.3. Proposition. *$\overline{\mathsf{P}}$ is a lax extension of the functor P to Rel.*

Proof. To show (E1) let $r : X \nrightarrow Y$ and $s : Y \nrightarrow Z$ then if $(\mathfrak{f}, \mathfrak{g}) \in \overline{\mathsf{P}}s \cdot \overline{\mathsf{P}}r$ there exists $\mathfrak{h}$ such that for all $\mu \in BY$, $\nu \in BZ$ and $\omega < \infty$: $\mathfrak{f}(r^\circ\mu \wedge \omega) \leqslant \mathfrak{h}(\mu)$ and $\mathfrak{h}(s^\circ\nu \wedge \omega) \leqslant \mathfrak{g}(\nu)$ and thus it follows from 5.2 that $\mathfrak{f}(r^\circ s^\circ \nu \wedge \omega) \leqslant \mathfrak{h}(s^\circ\nu \wedge \omega) \leqslant \mathfrak{g}(\nu)$ which implies that $(\mathfrak{f}, \mathfrak{g}) \in \overline{\mathsf{P}}(s \cdot r)$.

(E2) is verified analogously and (E3) holds by definition.

To show (E4) let $f : X \to Y$ be a function and suppose that $(\mathfrak{f}, \mathfrak{g}) \in \mathsf{P}f$ i.e. $\mathfrak{g} = \mathfrak{f}(- \cdot f)$. Then it follows that for any $\mu \in BY$, $\mathfrak{f}(f^\circ\mu) = \mathfrak{f}(\mu \cdot f) = \mathfrak{g}(\mu)$ and hence $(\mathfrak{f}, \mathfrak{g}) \in \overline{\mathsf{P}}r$. If $(\mathfrak{f}, \mathfrak{g}) \in (\mathsf{P}f)^\circ$ and $\nu \in BX$ then since $(f^\circ)^\circ(\nu) \cdot f \leqslant \nu$ we have $\mathfrak{f}((f^\circ)^\circ(\nu)) = \mathfrak{g}((f^\circ)^\circ(\nu) \cdot f) \leqslant \mathfrak{g}(\nu)$ and hence $(\mathfrak{f}, \mathfrak{g}) \in \overline{\mathsf{P}}f^\circ$. □

5.4. Proposition. *$\overline{\mathbb{P}}$ is a lax extension of the monad $\mathbb{P}$.*

Proof. That

$$\begin{tikzcd} X \arrow[r, "e_X"] \arrow[d, "r"'] \arrow[dr, phantom, "\leqslant"] & \mathsf{P}X \arrow[d, "\overline{\mathsf{P}}r"] \\ Y \arrow[r, "e_Y"'] & \mathsf{P}Y \end{tikzcd}$$

follows from the fact that if $(x, y) \in r$ then

$$\begin{aligned} \mathrm{ev}_x\big(r^\circ \wedge \omega\big) &= r^\circ\mu(x) \wedge \omega \\ &\leqslant \mu(y) = \mathrm{ev}_y(\mu) \end{aligned}$$

We next check that the multiplication too becomes op-lax.

To see that

$$\begin{tikzcd} \mathsf{P}^2X \arrow[r, "m_X"] \arrow[d, "\overline{\mathsf{P}}^2r"'] \arrow[dr, phantom, "\leqslant"] & \mathsf{P}X \arrow[d, "\overline{\mathsf{P}}r"] \\ \mathsf{P}^2Y \arrow[r, "m_Y"'] & \mathsf{P}Y \end{tikzcd}$$

holds let $(F, G) \in \overline{\mathsf{P}}^2r$, $\mu \in BY$ and $\omega < \infty$. Then we have

$$F\big((\overline{\mathsf{P}}r)^\circ\, \mathrm{ev}_\mu \wedge \omega\big) \leqslant G(\mathrm{ev}_\mu)$$

and if $\mathfrak{f} \in \mathsf{P}X$ and $\mathfrak{g} \in \mathsf{P}Y$ are such that $(\mathfrak{f}, \mathfrak{g}) \in \overline{\mathsf{P}}r$ then

$$\mathrm{ev}_{r^\circ\mu\wedge\omega}(\mathfrak{f}) = \mathfrak{f}\big(r^\circ\mu \wedge \omega\big) \leqslant \mathfrak{g}(\mu) \wedge \omega$$

Hence

$$\mathrm{ev}_{r^\circ\mu\wedge\omega}(\mathfrak{f}) \leqslant \inf_{(\mathfrak{f},\mathfrak{g})\in\overline{\mathsf{P}}r} \mathfrak{g}(\mu) \wedge \omega = (\overline{\mathsf{P}}r)^\circ\, \mathrm{ev}_\mu(\mathfrak{f}) \wedge \omega$$

and

$$\begin{aligned} m_X F\big(r^\circ\mu \wedge \omega\big) &= F(\mathrm{ev}_{r^\circ\mu\wedge\omega}) \\ &\leqslant F\big((\overline{\mathsf{P}}r)^\circ\, \mathrm{ev}_\mu(\mathfrak{f}) \wedge \omega\big) \\ &\leqslant G(\mathrm{ev}_\mu) \\ &= m_Y G(\mu) \qquad \square \end{aligned}$$

6. Initial lax extensions for submonads of $\mathbb{P}$

Making use of the results in the previous sections we are now in a position to extend all monads via initiality. We begin with the monad $\boldsymbol{\pi}$ and consider the morphism $a : \boldsymbol{\pi} \Rightarrow \mathbb{P}$. In order to do this we only need to define the action on relations. Let $r : X \nrightarrow Y$ be a relation, then put

$$\overline{\pi}r := a_Y^\circ \cdot \overline{\mathsf{P}}r \cdot a_X$$

which means that

$$(\mathfrak{F},\mathfrak{G})\in\overline{\pi}r \quad \text{if and only if} \quad \big(l(-)(\mathfrak{F}),l(-)(\mathfrak{G})\big)\in\overline{\mathsf{P}}r$$

It will turn out that this is exactly the lax extension of the prime functional ideal monad defined in [11] which we briefly recall. For a functional ideal $\mathfrak{G}$ on Y and a relation $r:X\nrightarrow Y$, $r^\circ\mathfrak{G}:=\{r^\circ\nu\wedge\omega \mid \nu\in\mathfrak{G} \text{ and } \omega<\infty\}^\sigma$ and then $\hat{\pi}r:\pi X\to\pi Y$ is the relation that contains all pairs $(\mathfrak{F},\mathfrak{G})$ satisfying $\forall\nu\in\mathfrak{G},\ \forall\omega<\infty$: $r^\circ\nu\wedge\omega\in\mathfrak{F}$ i.e.

$$(\mathfrak{F},\mathfrak{G})\in\hat{\pi}r \quad \text{if and only if} \quad r^\circ\mathfrak{G}\subseteq\mathfrak{F}$$

The unit and multiplication of this extension are given by the ones for the prime functional ideal monad.

6.1. Proposition. *For any relation $r:X\nrightarrow Y$: $\overline{\pi}r=\hat{\pi}r$.*

Proof. Let $\mathfrak{F}\in\pi X$ and $\mathfrak{G}\in\pi Y$. Suppose that $(\mathfrak{F},\mathfrak{G})\in\overline{\pi}r$. Let $\nu\in BY$, $\omega<\infty$ and suppose that $l(\nu)(\mathfrak{G})<\alpha$ which means that $\nu\in\mathfrak{G}\oplus\alpha$ and consequently also $r^\circ(\nu\ominus\alpha)\wedge\omega\in\mathfrak{F}$. Now for any $x\in X$ we have

$$\begin{aligned}\big(r^\circ\nu\wedge\omega\big)\ominus\alpha(x)&=\begin{cases}(\inf_{y\in r(x)}\nu(y)\wedge\omega)\ominus\alpha & r(x)\neq\emptyset\\ \omega\ominus\alpha & r(x)=\emptyset\end{cases}\\&\leqslant\begin{cases}(\inf_{y\in r(x)}\nu(y)\ominus\alpha)\wedge\omega & r(x)\neq\emptyset\\ \omega & r(x)=\emptyset\end{cases}\\&=r^\circ(\nu\ominus\alpha)\wedge\omega(x)\end{aligned}$$

Hence also $(r^\circ\nu\wedge\omega)\ominus\alpha\in\mathfrak{F}$ which implies that $l(r^\circ\nu)(\mathfrak{F})<\alpha$ and thus $l(r^\circ)(\mathfrak{F})\leqslant l(\nu)(\mathfrak{G})$.

Conversely suppose that $(\mathfrak{F},\mathfrak{G})\in\hat{\pi}r$, i.e. $r^\circ\mathfrak{G}\subseteq\mathfrak{F}$ then if $\nu\in\mathfrak{G}$ and $\omega<\infty$ it follows that

$$l\big(r^\circ\nu\wedge\omega\big)(\mathfrak{F})\leqslant l(\nu)(\mathfrak{G})=0$$

and thus for any $\omega<\infty$: $r^\circ\nu\wedge\omega\in\mathfrak{F}$, i.e. $(\mathfrak{F},\mathfrak{G})\in\overline{\pi}r$. $\square$

In the sequel we will denote the extension by $\overline{\boldsymbol{\pi}}:=(\overline{\pi},e,m)$.

6.2. Proposition. $\overline{\mathsf{P}}$ *and* $\overline{\pi}$ *satisfy* (IE).

Proof. We have to verify that for any $r:X\nrightarrow Y$ and with $\rho=a\circ a$, $\overline{\pi}^2r=\rho_Y^\circ\cdot\overline{\mathsf{P}}^2r\cdot\rho_X$. Suppose that $(\Phi,\Psi)\in\overline{\boldsymbol{\pi}}^2r$. Note that for any $\varphi\in B\mathsf{P}X$,

$$\begin{aligned}\rho_X\Phi(\varphi)&=a_{\mathsf{P}X}(\pi a_X)\Phi(\varphi)\\&=l(\varphi)\big(\{\gamma\in B\mathsf{P}X\mid\gamma\cdot a_X\in\Phi\}\big)\\&=\inf\big\{\alpha\ \big|\ (\varphi\ominus\alpha)\cdot a_X\in\Phi\big\}\end{aligned}$$

and analogously for $\lambda\in B\mathsf{P}Y$,

$$\rho_Y\Psi(\lambda)=\inf\big\{\alpha\ \big|\ (\lambda\ominus\alpha)\cdot a_Y\in\Psi\big\}$$

Hence, let now $\lambda\in B\mathsf{P}Y$, $\omega<\infty$ and let α be such that $\nu:=(\lambda\ominus\alpha)\cdot a_Y\in\Psi$. Then we have that $(\overline{\pi}r)^\circ\nu\wedge\omega\in\Phi$. For any $\mathfrak{F}\in\pi X$ we then have

$$\begin{aligned}\big((\overline{\mathsf{P}}r)^\circ\nu\big)l(-)(\mathfrak{F})&=\inf_{(l(-)(\mathfrak{F}),\mathfrak{f})\in\overline{\mathsf{P}}r}\lambda(\mathfrak{f})\wedge\omega\ominus\alpha\\&\leqslant\inf_{(l(-)(\mathfrak{F}),\mathfrak{f})\in\overline{\mathsf{P}}r}\lambda\ominus\alpha(\mathfrak{f})\wedge\omega\\&\leqslant\inf_{(l(-)(\mathfrak{F}),l(-)(\mathfrak{G}))\in\overline{\mathsf{P}}r}\lambda\ominus\alpha\big(l(-)(\mathfrak{G})\big)\wedge\omega\\&=\inf_{(\mathfrak{F},\mathfrak{G})\in\overline{\pi}r}\lambda\ominus\alpha\big(l(-)(\mathfrak{G})\big)\wedge\omega\\&=\big((\overline{\pi}r)^\circ\big((\lambda\ominus\alpha)\cdot a_Y\big)\wedge\omega\big)(\mathfrak{F})\\&=\big((\overline{\pi}r)^\circ\nu\wedge\omega\big)(\mathfrak{F})\end{aligned}$$

from which we can conclude that $((\overline{\mathsf{P}}r)^\circ\lambda\wedge\omega\ominus\alpha)\cdot a_X\in\Phi$ and that $\rho_X\Phi((\overline{\mathsf{P}}r)^\circ\lambda\wedge\omega)\leqslant\alpha$. By the arbitrariness of α this implies that for any $\lambda\in B\mathsf{P}Y$ and $\omega<\infty$,

$$\rho_X\Phi\big((\overline{\mathsf{P}}r)^\circ\lambda\wedge\omega\big)\leqslant\rho_Y\Psi(\lambda)$$

which in turn implies that $(\rho_X\Phi,\rho_Y\Psi)\in\overline{\mathsf{P}}^2r$.

Conversely, suppose that $(\rho_X\Phi,\rho_Y\Psi)\in\overline{\mathsf{P}}^2r$ and let $\nu\in\Psi$ and $\omega<\infty$. We extend $\nu\in B\pi Y$ in the following way

$$\lambda:\mathsf{P}Y\to\mathsf{P}:\mathfrak{f}\mapsto\begin{cases}\nu(\mathfrak{F}) & \mathfrak{f}=a_Y\mathfrak{F}\\ \sup\nu & a_Y^\circ\mathfrak{f}=\emptyset\end{cases}$$

Note that λ is well defined by injectivity of a_Y and that $\lambda\cdot a_X=\nu\in\Psi$. Further

$$\begin{aligned}\rho_X\Phi\big((\overline{\mathsf{P}}r)^\circ\lambda\wedge\omega\big)&\leqslant\rho_Y\Psi(\lambda)\\&=\inf\{\beta\mid\lambda\ominus\beta\cdot a_Y\in\Psi\}\\&=0\end{aligned}$$

and hence also

$$\inf\big\{\alpha\ \big|\ \big(\big((\overline{\mathsf{P}}r)^\circ\lambda\wedge\omega\big)\ominus\alpha\big)\cdot a_X\in\Phi\big\}=0$$

which implies that $((\overline{\mathsf{P}}r)^\circ\lambda\wedge\omega)\cdot a_X\in\Phi$. Now since for any $\mathfrak{F}$,

$$\begin{aligned}(\overline{\pi}r)^\circ\nu(\mathfrak{F})\wedge\omega&=\inf_{(\mathfrak{F},\mathfrak{G})\in\overline{\pi}r}\nu(\mathfrak{G})\wedge\omega\\&=\inf_{(l(-)(\mathfrak{F}),l(-)(\mathfrak{G}))\in\overline{\mathsf{P}}r}\nu(\mathfrak{G})\wedge\omega\\&\leqslant\inf_{(l(-)(\mathfrak{F}),\mathfrak{f})\in\overline{\mathsf{P}}r}\lambda(\mathfrak{f})\wedge\omega\\&=\big((\overline{\mathsf{P}}r)^\circ\lambda\wedge\omega\big)\cdot a_X(\mathfrak{F})\end{aligned}$$

this implies that $(\overline{\pi}r)^\circ\nu\wedge\omega\in\Phi$. This, in turn, implies that $(\Phi,\Psi)\in\overline{\boldsymbol{\pi}}^2r$. □

The results so far now allow us to deduce the following result.

6.3. Theorem. *$\overline{\boldsymbol{\pi}}$ is a lax extension of the monad $\boldsymbol{\pi}$.*

Again making use of initiality we now extend also the monads $\mathbb{D}$, $\mathbb{I}$ and $\boldsymbol{\beta}$. In order to do this we again only need to define the action on relations. Let $r:X\nrightarrow Y$ be a relation, then for D making use of the morphism $\delta:\mathbb{D}\Rightarrow\boldsymbol{\pi}$ this gives

$$\overline{\mathsf{D}}r:=\delta_Y^\circ\cdot\overline{\pi}r\cdot\delta_X$$

6.4. Proposition. *For any $r:X\nrightarrow Y$,*

$$\big(\omega_X(\dot{x})\oplus\alpha,\omega_Y(\dot{y})\oplus\beta\big)\in\overline{\mathsf{D}}r\quad\text{if and only if}\quad(x,y)\in r\text{ and }\beta\leqslant\alpha$$

Proof. By straightforward verification. □

In what follows we will use the following notation and convention. Given a functional ideal $\mathfrak{F}$ on a subset U of a given set V we denote by stack$(\mathfrak{F})$ the functional ideal "generated" on V, namely

$$\operatorname{stack}(\mathfrak{F}):=\big\{\nu\in BV\ \big|\ \exists\mu\in\mathfrak{F},\ \exists\tau<\infty\colon\ \nu\leqslant\mu^\tau\big\}$$

where

$$\mu^\tau(x):=\begin{cases}\mu(x) & x\in U\\ \tau & x\in V\setminus U\end{cases}$$

From the context it will be clear what U and V are.

6.5. Proposition. *$\overline{\pi}$ and $\overline{\mathsf{D}}$ satisfy* (IE).

Proof. Let

$$\Phi := \omega_{\mathsf{D}X}(\dot{\mathfrak{F}}) \oplus \alpha \quad \text{with} \quad \mathfrak{F} := \omega_X(\dot{x}) \oplus \gamma$$

and

$$\Psi := \omega_{\mathsf{D}Y}(\dot{\mathfrak{G}}) \oplus \beta \quad \text{with} \quad \mathfrak{G} := \omega_Y(\dot{y}) \oplus \delta$$

On the one hand, that $(\Phi, \Psi) \in \overline{\mathsf{D}}^2 r$ means that $(x, y) \in r$, $\beta \leqslant \alpha$ and $\delta \leqslant \gamma$. On the other hand, that $(\mathrm{stack}(\Phi), \mathrm{stack}(\Psi)) \in \overline{\pi}^2 r$ is easily seen to mean that for any $\omega < \infty$ and $\omega' < \infty$,

$$(\overline{\pi}^2 r)^\circ\big((\beta + \theta_{\{\mathfrak{G}\}}) \wedge \omega'\big) \wedge \omega(\mathfrak{F}) \leqslant \alpha$$

which in turn means that for all $\omega' < \infty$,

$$\inf_{(\mathfrak{F},\mathfrak{H}) \in \overline{\pi} r} (\beta + \theta_{\{\mathfrak{G}\}}) \wedge \omega'(\mathfrak{H}) \leqslant \alpha$$

This inequality can only hold for all $\omega' < \infty$ if $(\mathfrak{F}, \mathfrak{G}) \in \overline{\pi} r$ and it that case it means exactly $(x, y) \in r$, $\beta \leqslant \alpha$ and $\delta \leqslant \gamma$. □

This now again implies the following conclusion:

6.6. Theorem. *$\overline{\mathbb{D}}$ is a lax extension of the monad $\mathbb{D}$.*

Recall that for the usual lax extension of the ultrafilter functor to Rel [1] for any relation $r : X \nrightarrow Y$,

$$(\mathcal{U}, \mathcal{W}) \in \hat{\beta} r \quad \Leftrightarrow \quad \{r^{-1}(W) \mid W \in \mathcal{W}\} \subseteq \mathcal{U}$$

Again making use of initiality we now extend the monad $\boldsymbol{\beta}$ to Rel. In order to do this we only need to define the action on relations. Let $r : X \nrightarrow Y$ be a relation, then making use of the morphism $\omega : \boldsymbol{\beta} \Rightarrow \boldsymbol{\pi}$ we put

$$\overline{\beta} r := \omega_Y^\circ \cdot \overline{\pi} r \cdot \omega_X$$

6.7. Proposition. *For any relation $r : X \nrightarrow Y$: $\overline{\beta} r = \hat{\beta} r$.*

Proof. This is easily verified and we leave this to the reader. □

In the sequel we denote this extension $\overline{\beta}$.

6.8. Proposition. *$\overline{\pi}$ and $\overline{\beta}$ satisfy* (IE).

Proof. On the one hand, that $(\Xi, \Upsilon) \in \overline{\beta}^2 r$ means that for all $\mathcal{B} \in \Upsilon$,

$$\{\mathcal{U} \in \beta X \mid \exists\, \mathcal{W} \in \mathcal{B}\colon\ (\omega_X(\mathcal{U}), \omega_Y(\mathcal{W})) \in \overline{\pi} r\} \in \Xi$$

On the other hand, that $(\omega_X(\mathrm{stack}(\Xi)), \omega_Y(\mathrm{stack}(\Upsilon))) \in \overline{\pi}^2 r$ by definition means that for all $\mathcal{B} \in \Upsilon$,

$$\forall \omega < \infty\colon \quad (\overline{\pi} r)^\circ \theta_{\mathcal{B}} \wedge \omega \in \omega_X\big(\mathrm{stack}(\Xi)\big)$$

which in turn means for all $\mathcal{B} \in \Upsilon$,

$$Z\big((\overline{\pi} r)^\circ \theta_{\mathcal{B}} \wedge \omega\big) \in \mathrm{stack}(\Xi)$$

It can easily be seen and we leave it to the reader to verify that these two conditions are indeed equivalent. □

6.9. Theorem. *$\overline{\boldsymbol{\beta}}$ is a lax extension of the monad $\boldsymbol{\beta}$.*

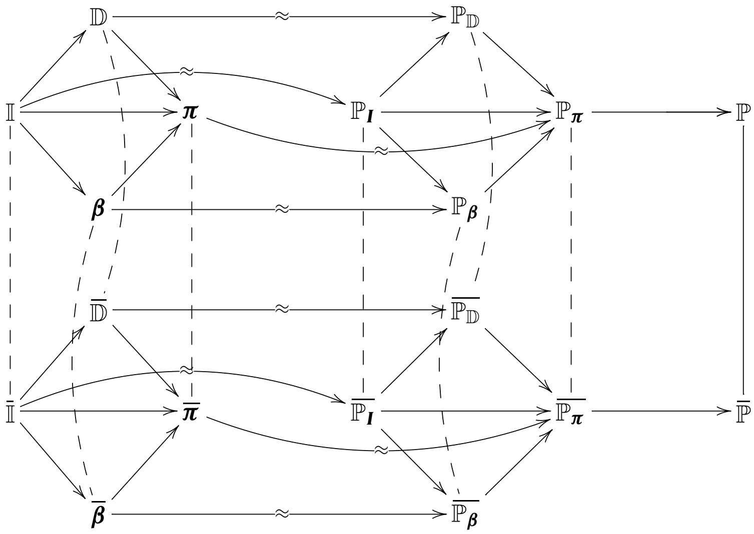
6.10. Theorem. *Both ω and ι become morphisms of lax extensions for respectively $\overline{\boldsymbol{\beta}}$ and $\overline{\boldsymbol{\pi}}$, where moreover ω is an initial morphism of lax extensions.*

Proof. For ω this follows at once from the formula $\overline{\beta} r = \omega_Y^\circ \cdot \overline{\pi} r \cdot \omega_X$. For ι we have $\overline{\pi} r \leqslant \iota_Y^\circ \cdot \overline{\beta} r \cdot \iota_X$ since for any relation $r : X \nrightarrow Y$ and $(\mathfrak{F}, \mathfrak{G}) \in \overline{\pi} r$ we have $(\iota_X(\mathfrak{F}), \iota_X(\mathfrak{G})) \in \overline{\beta} r$ by [11]. □

The case for $\mathbb{I}$ is a simple concatenation of the foregoing two and we leave the details to the reader.

6.11. Theorem. *$\overline{\mathbb{I}}$ is a lax extension of the monad $\mathbb{I}$.*

The results of the last two sections are captured in the following diagram where on the top we find all monads and corresponding monad morphisms. Only the monad $\mathbb{P}$ needs to be explicitly extended to $\overline{\mathbb{P}}$ (see the definition prior to 5.3). All other extensions, indicated by dotted arrows are derived from this one extension via initial lax extensions according. All these extensions are extensions of the monads.



7. Lax algebras

Recall from [2,4] that the category of lax algebras associated with a lax extension $\overline{\mathbb{T}} = (\overline{\mathsf{T}}, e, m)$ has as objects pairs (X, a) where $a : TX \nrightarrow X$ satisfies

$$\begin{tikzcd} X \arrow[r, "e_X"] \arrow[dr, "1_X"'] & \mathsf{T}X \arrow[d, "a", "\shortmid"{marking}] \arrow[dl, phantom, "\leqslant"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} \mathsf{T}^2X \arrow[r, "m_X"] \arrow[d, "\overline{\mathsf{T}}a"', "\shortmid"{marking}] \arrow[dr, phantom, "\leqslant"] & \mathsf{T}X \arrow[d, "a", "\shortmid"{marking}] \\ \mathsf{T}X \arrow[r, "a"', "\shortmid"{marking}] & X \end{tikzcd}$$

These conditions are respectively called the *reflexivity* and the *transitivity* condition. Morphisms from (X, a) to (Y, b) are functions $f : X \to Y$ satisfying:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, phantom, "\leqslant"] & Y \\ \mathsf{T}X \arrow[u, "a", "\shortmid"{marking}] \arrow[r, "\overline{\mathsf{T}}f"'] & \mathsf{T}Y \arrow[u, "b"', "\shortmid"{marking}] \end{tikzcd}$$

This category is denoted as $\mathsf{Alg}(\overline{\mathsf{T}}, e, m)$ or shortly $\mathsf{Alg}(\overline{\mathbb{T}})$. We first describe the category of lax algebras associated with the monad $\mathbb{P}$. By explicitly formulating reflexivity and transitivity we immediately obtain

7.1. Theorem. *The category $\mathsf{Alg}(\overline{\mathbb{P}})$ is concretely isomorphic to the category $\mathcal{X}$ with objects (X, a) where a is a relation on $\mathsf{P}X \times X$ satisfying*:

(1) $(\mathrm{ev}_x, x) \in a$,
(2) *for any $F \in \mathsf{P}^2X$, if there exists an $(\mathfrak{f}, x) \in a$ such that for any $\mu \in BX$ and $\omega < \infty$: $F(a^\circ\mu \wedge \omega) \leqslant \mathfrak{f}(\mu)$ then $(F \cdot \mathrm{ev}_-, x) \in a$,*

and with morphisms those functions $f : (X, a) \to (Y, b)$ that satisfy

$$(\mathfrak{f}, x) \in a \to \exists \mathfrak{g} \in \mathsf{P}Y, \quad \big(\mathfrak{g}, f(x)\big) \in b \quad \text{such that} \quad \mathfrak{f}(- \cdot f) \leqslant \mathfrak{g}$$

Restricting to $\mathbb{P}_\pi$ we get a similar characterization for its lax algebras. We are able to present a more elegant description based on the results in [11] describing $\mathsf{Alg}(\overline{\boldsymbol{\pi}})$ as the category of approach spaces with contractions and describing the objects in terms of convergence of prime functional ideals.

7.2. Theorem. *The category* $\mathsf{Alg}(\overline{\mathbb{P}_\pi})$ *and hence the category* App *of approach spaces is concretely isomorphic to the category* $\mathcal{X}$ *with objects* $(X, \to)$ *where* $\to$ *is a relation on* $\mathsf{P}_\pi X \times X$ *satisfying*:

(1) $\mathrm{ev}_x \to x$,
(2) $\mathfrak{g} \leqslant \mathfrak{f}$ *and* $\mathfrak{f} \to x \Rightarrow \mathfrak{g} \to x$,
(3) $\mathfrak{f}_z \to z$ *for all* $z \in X$ *and* $\mathfrak{f} \to x \Rightarrow \bigwedge_{\mu \in Z(\mathfrak{f})} \bigvee_{x \in X} \mathfrak{f}_z \ominus \mu(z) \to x$,

and with morphisms those functions $f : (X, \to) \longrightarrow (Y, \to)$ *that satisfy*

$$\mathfrak{f} \to x \quad \Rightarrow \quad \mathfrak{f}(- \cdot f) \to f(x)$$

Proof. We use the characterization of approach spaces given in [11] where it is shown that the category $\mathsf{Alg}(\overline{\boldsymbol{\pi}})$ has as objects sets equipped with a relation $\to$ on $\pi X \times X$ satisfying:

(C1) For every $x \in X : \omega_X(\dot{x}) \to x$.
(C2) If $\mathfrak{F}$ and $\mathfrak{G}$ are functional ideals, $\mathfrak{F} \subseteq \mathfrak{G}$ and $\mathfrak{F} \to x$ then $\mathfrak{G} \to x$.
(C3) If $\mathcal{S} = (\mathfrak{F}_z)_{z \in X}$ is a selection of functional ideals such that $\mathfrak{F}_z \to z$ for all $z \in X$ and $\mathfrak{F}$ is a functional ideal such that $\mathfrak{F} \to x$ then $\mathfrak{D}(\mathcal{S}, \mathfrak{F}) \to x$.

Morphisms functions $f : (X, \to) \longrightarrow (Y, \to)$ satisfying:

$$\mathfrak{F} \to x \quad \Rightarrow \quad \pi f(\mathfrak{F}) \to f(x)$$

Note that (C3) is an alternative form of the transitivity axiom where the operator $\mathfrak{D}$ is defined as

$$\mathfrak{D}(\mathcal{S}, \mathfrak{F}) := \bigvee_{\mathcal{A} \in \iota_X(\mathfrak{F})} \bigcap_{\mathfrak{U} \in \mathcal{A}} \mathfrak{U} \oplus c(\mathfrak{F})$$

Define the following concrete functor $F : \mathsf{Alg}(\overline{\boldsymbol{\pi}}) \to \mathcal{X}$. For an approach space X with its structure described in terms of convergence of prime functional ideals let $FX := (X, \to)$ where $\mathfrak{f} \to x$ if and only if $Z(\mathfrak{f})$ converges to x. That $\to$ satisfies (1) and (2) is clear. To see that it also satisfies (3) it suffices to remark from 3.7 that

$$Z\Bigl(\bigwedge_{\mu \in Z(\mathfrak{f})} \bigvee_{z \in X} \mathfrak{f}_z \ominus \mu(z)\Bigr) = \bigvee_{\mu \in Z(\mathfrak{f})} \bigcap_{z \in X} Z(\mathfrak{f}) \oplus \mu(z)$$

That F is concrete and full follows from the equality

$$Z\bigl(\mathfrak{f}(- \cdot h)\bigr) = \pi h\bigl(Z(\mathfrak{f})\bigr)$$

that it is injective on objects follows from the equality $Z(l(-)(\mathfrak{F})) = \mathfrak{F}$, and that it is surjective on objects finally follows from defining, for a given $(X, \to) \in \mathcal{X}$, $\mathfrak{F}$ to converge to x if and only if $l(-)(\mathfrak{F}) \to x$. □

We now first turn our attention to the lax algebras for $\overline{\mathbb{D}}$.

7.3. Theorem. *The category* $\mathsf{Alg}(\overline{\mathbb{D}})$ *has as objects pairs* (X, c) *where* $c : \mathsf{D}X \longrightarrow\!\!\!\!\!\!\!+\;\; X$ *satisfies*:

(D1) *For any* $x \in X$*:* $(\omega_X(\dot{x}), x) \in c$.
(D2) *If* $(\omega_X(\dot{x}) \oplus \beta, y) \in c$, $(\omega_X(\dot{y}) \oplus \alpha, z) \in c$ *then* $(\omega_X(\dot{x}) \oplus (\alpha + \beta), z) \in c$.
(D3) *If* $(\omega_X(\dot{y}) \oplus \gamma, z) \in c$ *and* $\gamma \leqslant \alpha$ *then* $(\omega_X(\dot{y}) \oplus \alpha, z) \in c$.

The morphisms are functions $f : (X, a) \to (Y, b)$ *satisfying*:

$$\bigl(\omega_X(\dot{z}) \oplus \alpha, x\bigr) \in a \quad \Rightarrow \quad \bigl(\omega_Y\bigl(f(\dot{z})\bigr) \oplus \alpha, f(x)\bigr) \in b$$

Proof. Immediate from the definitions. □

The conditions in the foregoing theorem have some noteworthy alternatives. Conditions (D2) and (D3) can be combined into: $(\omega_X(\dot{x}) \oplus \beta, y) \in c$, $(\omega_X(\dot{y}) \oplus \gamma, z) \in c$ and $\gamma \leqslant \alpha$ then $(\omega_X(\dot{x}) \oplus (\alpha + \beta), z) \in c$.

Further (D2) has the following equivalent form: for any selection $\mathcal{S} := (x_z, \alpha_z)_z$ in $X \times [0, \infty]$, if $(\omega_X(\dot{x}_z) \oplus \alpha_z, z) \in c$ for all z and $(\omega_X(\dot{y}) \oplus \alpha, x) \in c$ then $\mathfrak{D}(\mathcal{S}, \omega_X(\dot{y}) \oplus \alpha) \in c$.

This follows, among other things, from the observation that for any selection $\mathcal{S} := (x_z, \alpha_z)_z$ in $X \times [0, \infty]$: $\mathfrak{D}(\mathcal{S}, \omega(\dot{y}) \oplus \alpha) = \omega(\dot{x}_y) \oplus (\alpha_y + \alpha)$.

By [4] we have that $\delta : \mathbb{D} \Rightarrow \boldsymbol{\pi}$ induces a so called algebraic functor

$$F_\delta : \mathsf{Alg}(\overline{\boldsymbol{\pi}}) \to \mathsf{Alg}(\overline{\mathbb{D}})$$

derived from the morphism of lax extension δ. Remark however that F_δ is not surjective on objects. On any given set X let

$$\big(\omega_X(\dot{x}) \oplus \epsilon, y\big) \in c \quad \Leftrightarrow \quad \begin{cases} x \neq y & \epsilon > 0 \\ x = y & \epsilon \geqslant 0 \end{cases}$$

Then (X, c) is a lax algebra for $\overline{\mathbb{D}}$ but in view of results of [11] c cannot be obtained as $a \cdot \delta_X$ for some lax algebra a for $\overline{\boldsymbol{\pi}}$.

The crucial property involved here is what we call *saturatedness*.

7.4. Definition. A lax algebra (X, c) for $\overline{\mathbb{D}}$ is said to be *saturated* if

$$\big(\forall \alpha > \beta\colon \big(\omega_X(\dot{y}) \oplus \alpha, x\big) \in c\big) \quad \Rightarrow \quad \big(\omega_X(\dot{y}) \oplus \beta, x\big) \in c$$

From [11] it follows that all lax algebras $(X, a \cdot \delta_X)$ for some lax algebra (X, a) in $\mathsf{Alg}(\overline{\boldsymbol{\pi}})$ are saturated. We now determine the image of F_δ. We recall that $pq\,\mathsf{Met}^\infty$ stands for the topological construct consisting of all pseudo-quasi-metric spaces equipped with non-expansive maps as morphisms. Consider

$$G : pq\,\mathsf{Met}^\infty \to \mathsf{Alg}(\overline{\mathbb{D}})$$

defined on objects by $G(X, \varphi) := (X, c_\varphi)$ where $(\omega_X(\dot{z}) \oplus \alpha, x) \in c_\varphi$ if and only if $\varphi(x, z) \leqslant \alpha$. Then (X, c_φ) is indeed a lax algebra and G is a concrete embedding. Further let

$$F : \mathsf{Alg}(\overline{\mathbb{D}}) \to pq\,\mathsf{Met}^\infty$$

be defined on objects by $F(X, c) := (X, \varphi_c)$ where

$$\varphi_c(x, y) := \inf\big\{\epsilon \;\big|\; \big(\omega_X(\dot{y}) \oplus \epsilon, x\big) \in c\big\}$$

then (X, φ_c) is an ∞pq-metric space and F defines a concrete functor. Moreover we have $F \cdot G = 1_{pq\mathsf{Met}^\infty}$ and $G \cdot F \geqslant 1_{\mathsf{Alg}(\overline{\mathbb{D}})}$.

7.5. Theorem. *For a lax algebra (X, c) in $\mathsf{Alg}(\overline{\mathbb{D}})$ the following are equivalent*:

(1) (X, c) *is saturated.*
(2) (X, c) *comes from a pseudo-quasi-metric (i.e. $c = c_{\varphi_c}$).*
(3) (X, c) *is in the image of F_δ.*

Proof. Suppose c is saturated. If $(\omega(\dot{y}) \oplus \beta, x) \notin c$ then for some $\alpha > \beta$, $\omega(\dot{y}) \oplus \alpha \notin c$. This implies that $\varphi_c(x, y) \geqslant \alpha$ and therefore also $\varphi_c(x, y) > \beta$ and $(\omega(\dot{y}) \oplus \beta, x) \notin c_{\varphi_c}$.

Suppose that $c = c_{\varphi_c}$. We then define a relation a as follows

$$(\mathfrak{F}, x) \in a \quad \Leftrightarrow \quad \cap\big\{\mathfrak{G} \in \overline{\mathsf{D}}X \;\big|\; (\mathfrak{G}, x) \in c\big\} \subseteq \mathfrak{F}$$

which also means

$$(\mathfrak{F}, x) \in a \quad \Leftrightarrow \quad \varphi_c(x, -) \in \mathfrak{F}$$

We show that (X, a) is a lax algebra in $\mathsf{Alg}(\overline{\boldsymbol{\pi}})$. Using the characterization which we recalled in the proof of 7.2, the only non-trivial part is (C3). Let $(\mathfrak{F}_z, z) \in a$ for $z \in X$ and let $(\mathfrak{F}, x) \in a$. This implies that $\varphi_c(z, -) \in \mathfrak{F}_z$ for $z \in X$ and $\varphi_c(x, -) \in \mathfrak{F}$. For $\epsilon > 0$ put

$$F_\epsilon := \big\{\varphi_c(x, -) < c(\mathfrak{F}) + \epsilon\big\}$$

then

$$\varphi_c(x, -) \leqslant \inf_{z \in F_\epsilon} \varphi_c(z, -) + c(\mathfrak{F}) + \epsilon$$

and by the arbitrariness of $\epsilon > 0$ this proves that

$$\varphi_c(x, -) \in \bigvee_{F \in \iota_X(\mathfrak{F})} \bigcap_{z \in F} \mathfrak{F}_z \oplus c(\mathfrak{F})$$

Finally, suppose that (X, c) is a lax algebra in $\mathsf{Alg}(\overline{\mathbb{D}})$ and $c = a \cdot \delta_X$ for some lax algebra (X, a) in $\mathsf{Alg}(\overline{\boldsymbol{\pi}})$. Then by 8.2 in [11] c is saturated. $\square$

7.6. Corollary. *The construct* $\mathsf{Alg}_s(\overline{\mathbb{D}})$ *with objects all saturated lax algebras in* $\mathsf{Alg}(\overline{\mathbb{D}})$ *is a reflective subconstruct of* $\mathsf{Alg}(\overline{\mathbb{D}})$ *which is isomorphic to* $pq\,\mathsf{Met}^{\infty}$.

7.7. Theorem. *The category* $\mathsf{Alg}(\overline{\mathbb{P}}_{\mathsf{D}})$ *is concretely isomorphic to the category* $\mathcal{Y}$ *with objects* $(X, \to)$ *where* $\to$ *is a relation on* $\mathsf{D}X \times X$ *satisfying*:

(1) $\mathrm{ev}_x \to x$,
(2) $\mathfrak{g} \leqslant \mathfrak{f}$ *and* $\mathfrak{f} \to x \Rightarrow \mathfrak{g} \to x$,
(3) $\mathfrak{f} \to y$ *and* $\mathrm{ev}_y \ominus \gamma \to z \Rightarrow \mathfrak{f} \ominus \gamma \to z$,

and with morphisms those functions $f : (X, \to) \longrightarrow (Y, \to)$ *that satisfy*

$$\mathfrak{f} \to x \quad \Rightarrow \quad \mathfrak{f}(- \cdot f) \to f(x)$$

Among these objects, those that satisfy the following supplementary condition

(4) $(\forall \alpha > \beta\colon \mathfrak{f} \ominus \alpha \to x) \Rightarrow \mathfrak{f} \ominus \beta \to x$,

constitute a concretely reflective subconstruct isomorphic to $pq\,\mathsf{Met}^{\infty}$.

Proof. Immediate from 7.3 and 7.6. □

We now turn our attention to $\overline{\boldsymbol{\beta}}$. The category $\mathsf{Alg}(\overline{\boldsymbol{\beta}})$ has been characterized by Barr in [1] as the category of topological spaces and continuous maps. This characterization is by means of convergence of ultrafilters which, in an adapted and more tangible form, is the following.

7.8. Theorem. *The category* $\mathsf{Alg}(\overline{\boldsymbol{\beta}})$ *has as objects sets equipped with a relation* $\to$ *on* $\beta X \times X$ *satisfying*:

(T1) *For every* $x \in X : \dot{x} \to x$.
(T2) *If* $\mathcal{S} := (\mathcal{U}_x)_{x \in X}$ *is a selection in* βX *such that for every* $z \in X, \mathcal{U}_z \to z$ *and* $\mathcal{U} \in \beta X$ *such that* $\mathcal{U} \to x$ *then* $\mathcal{D}(\mathcal{U}, \mathcal{S}) \to x$.

And morphisms those functions $(X, \to) \longrightarrow (Y, \to)$ *that satisfy*

$$\mathcal{U} \to x \quad \Rightarrow \quad \beta f(\mathcal{U}) \to f(x)$$

and as such is concretely isomorphic to Top.

Here again we will obtain an isomorphic characterization. First we investigate the algebraic functors involved.

$$F_\omega : \mathsf{Alg}(\overline{\boldsymbol{\pi}}) \to \mathsf{Alg}(\overline{\boldsymbol{\beta}}) : (X, b) \mapsto (X, b \cdot \omega_X)$$

and

$$F_\iota : \mathsf{Alg}(\overline{\boldsymbol{\beta}}) \to \mathsf{Alg}(\overline{\boldsymbol{\pi}}) : (X, a) \mapsto (X, a \cdot \iota_X)$$

derived from the morphisms of lax extensions ω and ι.

7.9. Proposition. *There is an induced adjunction* $F_\omega \dashv F_\iota$ *and* $F_\iota \cdot F_\omega$ *coincides with the coreflector from approach spaces to topological approach spaces.*

Proof. Apply 3.12 in [15] to get the adjunction. We have

$$F_\omega \cdot F_\iota a = a \cdot \iota_X \cdot \omega_X = a \quad \text{and} \quad F_\iota \cdot F_\omega b = b \cdot \omega_X \cdot \iota_X \leqslant b$$

For the approach space corresponding to b the lax algebra $b \cdot \omega_X \cdot \iota_X$ defines the topological coreflection. □

7.10. Theorem. *The category* $\mathsf{Alg}(\overline{\mathbb{P}}_\beta)$ *and hence the category* Top *of topological spaces is concretely isomorphic to the category* $\mathcal{Z}$ *with objects* $(X, \to)$ *where* $\to$ *is a relation on* $\mathsf{P}_\beta X \times X$ *satisfying*:

(1) $\mathrm{ev}_x \to x$,
(2) $\mathfrak{f}_z \to z$ *for all* $z \in X$ *and* $\mathfrak{f} \to x \Rightarrow \bigwedge_{\mu \in Z(\mathfrak{f}), Z(\mu) \neq \emptyset} \vee_{z \in Z(\mu)} \mathfrak{f}_z \to x$,

and with morphisms those functions $f:(X,\to)\longrightarrow(Y,\to)$ *that satisfy*

$$\mathfrak{f}\to x \quad\Rightarrow\quad \mathfrak{f}(-\cdot f)\to f(x)$$

Proof. Immediate from 7.8. □

Finally $\mathsf{Alg}(\bar{\mathbb{I}})$ is concretely isomorphic to the category Ord of pre-ordered spaces and order-preserving maps [5]. The following gives an alternative characterization.

7.11. Theorem. *The category* $\mathsf{Alg}(\overline{\mathbb{P}}_{\mathsf{I}})$ *and hence the category* Ord *of pre-ordered spaces is concretely isomorphic to the category* $\mathcal{O}$ *with objects* $(X,\to)$ *where* $\to$ *is a relation on* $\mathsf{P}_I X\times X$ *satisfying*:

(1) $\mathrm{ev}_x\to x$,
(2) $\mathrm{ev}_x\to y$ *and* $\mathrm{ev}_y\to z\Rightarrow \mathrm{ev}_x\to z$,

and with morphisms those functions $f:(X,\to)\longrightarrow(Y,\to)$ *that satisfy*

$$\mathrm{ev}_y\to x \quad\Rightarrow\quad \mathrm{ev}_{f(y)}\to f(x)$$

Proof. Immediate from 3.20 and 7.7. □

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